Relevant Logic

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The earliest attempts to devise a relevance logic that avoided the problem of explosion centered on the conditional.

FDE, however, has no conditional operator, or a very weak one.

If we add one by defining $A \supset B \iff \neg A \lor B$, we get something that does not validate modus ponens (since it corresponds to disjunctive syllogism, given that definition).
One strategy is to let the added strength desired come from a non-monotonic consequence relation defined on the basis of FDE.

But most relevance logicians have agreed with other conditional theorists that the conditional is an intensional operator that cannot be defined in any simple way in terms of other operators.
Conditionals

- There are two strategies for adding conditionals. The first, pioneered by an early generation of researchers such as Anderson, Belnap, and Routley, is to approximate as closely as possible the material conditional or the strict conditional, leading to fairly strong logics such as B, R, and E.
- The central motivation is to characterize the logic of entailment.
The second, fully willing to embrace paraconsistency and inspired partly by Lewis and others on the conditionals, prefers weaker logics.
Criteria

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  - Another, primary for David Lewis, Robert Stalnaker, and other writers on counterfactuals, is giving a semantics for natural language conditionals.
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- One, primary for Russell and Whitehead, is formalizing reasoning in mathematics and the sciences.
- Another, primary for David Lewis, Robert Stalnaker, and other writers on counterfactuals, is giving a semantics for natural language conditionals.
- A third, primary for C. I. Lewis, Anderson, and Belnap, is providing a theory of entailment, thinking of the conditional as correlated with the logical notion of implication or entailment.
These differing goals lead logicians to take different inference patterns as paradigmatic and to disagree on particular inference patterns such as transitivity, contraposition, the deduction theorem, or even modus ponens.
We therefore add a conditional operator $\rightarrow$.

Let a relational interpretation $< W, \rho >$ consist of a set of worlds and a relation between world-proposition pairs and truth values.

For each $w \in W$, $\rho_w$ relates propositional parameters to $\{0, 1\}$.

We keep the truth clauses for negation, conjunction, and disjunction as they were for FDE, but relativized to worlds.
Truth Conditions

For the conditional:

\[ A \rightarrow B_{\rho_w}1 \iff \forall w' \in W \,(A_{\rho_w}1 \Rightarrow B_{\rho_w}1) \]

\[ A \rightarrow B_{\rho_w}0 \iff \exists w' \in W \,(A_{\rho_w}1 \land B_{\rho_w}0) \]
The logic K4 results from defining entailment in the natural way:

\[ \Sigma \models A \iff \forall < W, \rho > \forall w \in W (\Sigma \rho w 1 \Rightarrow A \rho w 1) \]
To obtain tableaux for K4, we need to index nodes to worlds.

So, a node will have the form $A, +i$ or $A, −i$, to indicate that $A$ relates to, or fails to relate to, truth at $w_i$.

A tableau begins with $\Sigma, +0; A, −0$. Branches close if they have formulas of the form $A, +i; A, −i$.

Rules for extensional connectives are the same as for FDE, but with world indices.
Tableau Rules

\[ A \rightarrow B, +i \]
\[ A, -j \quad B, +j \]

\[ A \rightarrow B, -i \]
\[ A, +j \]
\[ B, -j \]
\[ j \text{ new} \]
Tableau Rules

\[ \neg (A \rightarrow B), +i \]
\[ \mid \]
\[ A, +j \]
\[ \neg B, +j \]
\[ j \text{ new} \]

\[ \neg (A \rightarrow B), -i \]
\[ \mid \]
\[ A, -j \]
\[ \neg B, -j \]
Tableaux

- Open branches determine countermodels; $p, +i$ yields $p_{\rho_{w_i}1}$, and $\neg p, +i$ yields $p_{\rho_{w_i}0}$.
- The tableau system is sound and complete with respect to the relational semantics.
What is the conditional of K4 like? It obeys transitivity and strengthening of the antecedent, but not contraposition. It does not fall prey to the paradoxes of strict implication.

Nevertheless, $\models A \Rightarrow \models B \rightarrow A$.

So, in particular, $\models p \rightarrow (q \rightarrow q)$. This seems to violate the spirit of relevance.

But we cannot come up with a countermodel to it so long as every world validates $q \rightarrow q$. 
Note: The truth conditions for the conditional in K4 imply that $A \rightarrow B$ is true whenever $A \models_{FDE} B$.

So, there are many tautologies in K4, even though there are none at all in FDE.
Nonnormal Worlds

To devise a countermodel, therefore, we must use non-normal worlds.

Let an interpretation be a structure $< W, N, \rho >$, where $N \subseteq W$.

For every world $w \in W$, $\rho_w$ relates propositional parameters to truth values.

For every non-normal world $w \in W - N$, $\rho_w$ also relates conditionals to truth values.

At non-normal worlds, therefore, truth values of conditionals are not determined recursively.
If we define implication in terms of truth preservation at all normal worlds of all models, we obtain the logic N4:

\[ \Sigma \models A \iff \forall <W, N, \rho> \forall w \in N(\Sigma \rho_w 1 \Rightarrow A\rho_w 1) \]
Tableaux for N4 are exactly as for K4, except that rules for conditionals apply only at world 0.

Recall that when we begin a tableau in a non-normal modal logic, we assume that there is a *normal* world at which the premises are all true and the conclusion is not true.

So, we can assume world 0 to be normal. Truth values of conditional statements are thus determined recursively there.
• Beyond 0, however, we have no assurance that our worlds are normal, so we have no way to justify applying those rules.

• (If we strengthened the logic by adding constraints of the kind that produce S6, S7, S8, etc., from N, and add modal operators to the language, then we might have ways of showing that other worlds are normal; they might, for example, be $\Box$-inhabited.)
Open branches determine countermodels as for K4, except that world 0 is the only normal world, and we treat conditionals at other worlds as if they were propositional parameters.
Since all K4 interpretations are N4 interpretations (specifically, those in which there are no non-normal worlds), N4 \subset K4.
Nonnormal Worlds

- Non-normal worlds function as if they had access to logically impossible worlds, worlds in which the laws of logic fail or are different.

- This seems needed to account for sentences such as *If intuitionistic logic were correct, the law of double negation would fail* and *If intuitionistic logic were correct, the law of identity would fail*.

- The former seems true, the latter false.
Modus Ponens

- Note that, in non-normal worlds, modus ponens may fail: we may have $A$ and $A \rightarrow B$ but not $B$.
- This is not true of all inferences, but only those involving the conditional.
Also, note that, in N4, $p \rightarrow (q \rightarrow q)$ fails, but $p \models q \rightarrow q$.

The deduction theorem thus fails in N4.

Evidently $p \rightarrow (p \rightarrow p)$ also fails, even though it satisfies a variable sharing constraint.

N4 thus differs from the result of adopting K4 or for that matter T, S4, or S5 and simply throwing out inferences that violate the variable sharing constraint.

(Logics that operate in that way are known as *filter logics*.)
We have seen four semantics for FDE: Belnap’s four-valued semantics, Dunn’s relational semantics, Routley’s star semantics, and my three-valued semantics.

They are all equivalent; they yield exactly the same logic.

That is not true once we add a conditional. Routley semantics for the conditional produces a different logic from K4 or N4.
Let \( \langle W, *, v \rangle \) be a Routley interpretation, and define truth for conditional statements as follows:

\[
v_w(A \rightarrow B) = 1 \iff \forall w' \in W(v_{w'}(A) = 1 \Rightarrow v_{w'}(B) = 1)
\]

The logic that results, \( K^* \), is not equivalent to \( K4 \).
K* Tableaux

- To obtain tableaux for K*, start with tableau rules for FDE on a Routley interpretation.
- Nodes have the forms $A, +x$ or $A, -x$, where $x$ is either $i$ or $i^\ast$.
- Branches close if they contain nodes $A, +x$ and $A, -x$ for some $A$.
- The initial list is $\Sigma, +0, A, -0$. Let $x$ be either $i$ or $i^\ast$, and $\bar{x}$ be the other.
- The rules for conjunction and disjunction are what you would expect.
Tableau Rules

\[
\begin{align*}
A \lor B, +x & \\
A, +x & \quad B, +x
\end{align*}
\]

\[
\begin{align*}
A \lor B, -x & \\
A, -x & \\
B, -x
\end{align*}
\]
Tableau Rules

\[
A \land B, +x \\
\quad | \\
A, +x \\
B, +x
\]

\[
A \land B, -x \\
\quad | \\
A, -x \\
B, -x
\]
Tableau Rules

\[ \neg A, +x \quad \neg A, -x \]
\[ \quad | \quad \quad | \]
\[ A, -\bar{x} \quad A, +\bar{x} \]
Tableau Rules

\[ A \rightarrow B, +x \]
\[ A, -y \quad B, +y \]

\[ A \rightarrow B, -x \]
\[ A, +j \]
\[ B, -j \]
\[ j \text{ new} \]
Here $x$ is either $i$ or $i^*$ and $y$ is $j$ or $j^*$, where at least one is on the branch.

So, if either $j$ or $j^*$ occurs on the branch, we must apply the rule to both.

Open branches determine countermodels: if $p, +x$ occurs, $v_{w_x}(p) = 1$; if $p, -x$ occurs, $v_{w_x}(p) = 0$. 
It remains true that $\models p \rightarrow (q \rightarrow q)$.

To change this, add non-normal worlds, letting an interpretation be a structure $< W, N, *, v >$, where $N \subseteq W$, $w** = w$, and $v$ assigns a truth value to every parameter at each world, and every conditional at each non-normal world.

Truth conditions for conditionals apply only at normal worlds; at non-normal worlds they are given directly.
Validity is truth preservation at normal worlds.
The result is N*.
To obtain tableaux, use the rules of K*, but apply the conditional rules only at world 0.
The star semantics and the relational semantics are not equivalent once we add conditionals.

K* and N* make contraposition valid; K4 and N4 do not.

The latter validate $p \land \neg q \models \neg(p \rightarrow q)$; the former don’t.
Which of these logics obeys the variable sharing constraint?

- K4 and K* do not, for they validate $\models p \to (q \to q)$.
- But N4 and N* do. They have a better right to be considered relevant logics.
For N4: Say $A$ and $B$ have no propositional parameters in common.

Let $< W, N, \rho >$ be such that $W = \{w_0, w_1\}$, $N = \{w_0\}$, $D\rho_{w_1} 1$ and $D\rho_{w_1} 0$ for all parameters or conditionals $D$ in $A$, and neither for all parameters or conditionals $D$ in $B$.

It follows that $A\rho_{w_1} 1$ and $A\rho_{w_1} 0$ but neither $B\rho_{w_1} 1$ nor $B\rho_{w_1} 0$.

(One can prove this by induction, but it’s easier to think of the FDE four-valued truth tables: if all components get $n$, so do compounds, and similarly for $b$.)

So, $A$ is true at $w_1$ and $B$ isn’t; thus, at $w_0$, $A \rightarrow B$ fails.
These logics are very weak.

We can devise stronger logics by using another way of interpreting relevance logics, closely linked to Kripke's idea for modal semantics in general.

Recall that, to interpret unary operators □ and ◊, we introduced a binary accessibility relation.
To interpret a binary operator, →, Routley and Meyer decided to introduce a ternary operator.

$R_{xyz}$ means that, for all $A$ and $B$, if $A \rightarrow B$ is true at $x$, and $A$ is true at $y$, then $B$ is true at $z$.

This, like the Routley star, has been accused of being nothing more than a technical trick.
But it can be given interesting philosophical interpretations.

Perhaps the most obvious is simply this. What does it mean to say that $A \rightarrow B$ is true at $w$? We might imagine conducting an experiment, making it true that $A$ (thus moving to world $y$ in which the experiment is performed).

We then see whether $B$ occurs, examining the result of our experiment on condition that no interfering factors were present.
Routley Semantics for the Conditional

Ternary Accessibility

Extensions of B: R

Adding a Better Conditional

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That resultant circumstance may be that of starting the experiment, bringing it about that $A$, but perhaps not; call it $z$.

Normally, $z = y$; the experimental circumstance is just the resulting circumstance.

That will be true if the connection is logical.

But, more generally, temporal or interfering factors may lead them to deviate from each other.
More generally, think of the conditional $A \rightarrow B$ as asserted in a background situation $s_0$.

The “If $A$” part invokes a situation $s_1$ in which the antecedent is fulfilled, or at least tested.

We then check a resultant situation $s_2$ to see whether the consequent $B$ is fulfilled there.
Routley semantics uses worlds rather than situations, and indeed, it uses worlds in the standard sense, which contain no gaps and gluts.

But we saw earlier that we could think of the star as swapping gaps and gluts. The star function has the effect of pairing worlds so that world pairs linked by the star relation act as situations, which allow gaps, but that might also contain gluts.
Star Worlds

- To see how this works, recall that the Routley semantics counts $\neg p$ true at $w$ if $p$ is false, not at $w$ itself, but at $w^*$.  
- Consider two worlds $w$ and $w^*$ making the assignments:

<table>
<thead>
<tr>
<th></th>
<th>$w$</th>
<th>$w^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\neg p$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$q$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\neg q$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$r$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\neg r$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$s$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\neg s$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Star Worlds

- The pair agrees on \( p \) and \( q \), but disagrees on \( r \) and \( s \).
- This has the effect that \( w \) counts both \( r \) and \( \neg r \) true, while \( w^* \) counts both \( s \) and \( \neg s \) true.
- So, \( w \) gives \( r \) a glut and \( s \) a gap; \( w^* \) gives \( r \) a gap and \( s \) a glut.
- In that sense, the star function swaps gaps and gluts.
- Though each world assigns each propositional parameter one and only one truth value, the semantics admits gaps and gluts by relating pairs of worlds.
Another way of construing the ternary relation is information-theoretic, in terms of aggregating information in a way suggested by the way Ramsey and others have thought about conditionals.

- $R_{xyz}$ means that $z$ contains the information in $x$ and $y$ pooled together.
- The information of circumstance $x$, that is, supplemented by the information in $y$, is contained in $z$.
- This seems to suggest, however, that $B \rightarrow (A \rightarrow A)$ should be valid.
- No relevance logic could accept that result.
We might instead think of some situations as conduits for information—information channels, perhaps—and construe $Rxyz$ as meaning that $x$ carries information $y$ to $z$. 
Ramsey Test

- “If two people are arguing ‘If $p$, will $q$?’ and are both in doubt as to $p$, they are adding $p$ hypothetically to their stock of knowledge and arguing on that basis about $q$.”
- $x$: current stock of knowledge, current information state
- $y$: $x$ supplemented hypothetically with $p$
- $z$: resulting state—containing the information that $q$ if the conditional is true
Routley Semantics for the Conditional
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Relevant Logic
GOT AN A ON A TEST
MADE BY RAMSEY
Let a ternary interpretation be a structure \( < W, N, R, *, v > \), where \( W \) is a set of worlds, \( N \subseteq W \) a set of normal worlds, \( R \subseteq W \times W \times W \), \( * \) a function from worlds to worlds such that \( w^{**} = w \), and \( v \) a function from propositional parameters to truth values at each world, and from conditionals to truth values at each non-normal world.
Truth Conditions

- Truth conditions for connectives are those of N*, with one exception.
- At normal worlds $w$,

$$v_w(A \rightarrow B) = 1 \iff \forall x \in W (v_x(A) = 1 \Rightarrow v_x(B) = 1)$$

But if $w$ is non-normal,

$$v_w(A \rightarrow B) = 1 \iff \forall x, y \in W (Rwxy \Rightarrow (v_x(A) = 1 \Rightarrow v_y(B) = 1))$$

- Validity is truth preservation over normal worlds.
- The logic that results: B.
Any $K^*$ interpretation is a $B$-interpretation in which all worlds are normal.

So, $B \subseteq K^*$. Any $B$-interpretation, moreover, is equivalent to an $N^*$-interpretation, so $N^* \subseteq B$.

$B$ is thus between $N^*$ and $K^*$ in strength.
Having two truth conditions for conditionals is inelegant, and also suggests that the conditional itself has no univocal meaning.

But we might think of $R$ as defined at normal worlds by a normality condition: if $w$ is normal, $Rwxy \Rightarrow x = y$ (which is equivalent to $Rwxy \Rightarrow x = y$ and $Rwxw$).

Now, at all worlds,

$$v_w(A \rightarrow B) = 1 \iff \forall x, y \in W(Rwxy \Rightarrow (v_x(A) = 1 \Rightarrow v_y(B) = 1)).$$
Tableau Rules

Tableaux for B are like those for N*, but nodes may take the form $rxyz$, and rules for the conditional are:

\[
A \rightarrow B, +x \\
\text{rxyz} \\
A, -y \quad B, +z
\]

\[
A \rightarrow B, -x \\
\quad \text{rxyz} \\
\quad \text{rxjk} \\
A, +j \\
B, -k \\
j, k \text{ new}
\]
Tableaux

- If $x$ is 0, $j = k$. Also, $r0xx$.
- Countermodels may be read off branches as in N*, but now we have information about accessibility as well.
Valid in B and N*

\[
\begin{align*}
A & \rightarrow A \\
A & \rightarrow (A \lor B) \\
(A \land B) & \rightarrow A \\
\neg\neg A & \rightarrow A \\
A, A & \rightarrow B \not\models B \\
A & \rightarrow \neg B \not\models B \rightarrow \neg A
\end{align*}
\]
B validates some principles that are not valid in N*:

**Consequent conjunction**: \((A \rightarrow B) \land (A \rightarrow C) \models A \rightarrow (B \land C)\)

**Disjunction**: \((A \rightarrow C) \land (B \rightarrow C) \models (A \lor B) \rightarrow C\)

**Prefixing**: \(A \rightarrow B \models (C \rightarrow A) \rightarrow (C \rightarrow B)\)

**Suffixing**: \(A \rightarrow B \models (B \rightarrow C) \rightarrow (A \rightarrow C)\)

Because of the last two, it is known as an affixing relevant logic.
The most important extension of B is a logic known as R, which forms the foundation of much of Anderson and Belnap’s work on relevance logic.

R results from imposing four constraints on the ternary accessibility relation.

(Each, of course, may be added independently of the others, so R is one of 15 extensions of B that might be defined in terms of just these constraints.)

R is a complex logic; Urquhart proved that it is undecidable.
For all worlds $a$, $b$, $c$, and $d$:

$$Rabc \Rightarrow Rac^* b^*$$

$$\exists x \in W(Rab x \land Rxc d) \Rightarrow \exists y \in W(Racy \land Rby d)$$

$$\exists x \in W(Rab x \land Rxc d) \Rightarrow \exists y \in W(Rbc y \land Ray d)$$

$$Rabc \Rightarrow \exists x \in W(Rab x \land Rxbc)$$
Tableaux

- Corresponding to each of these is a tableau rule. But tableaux are extremely complicated!
- The above constraints validate, respectively,

  \[
  \text{Contraposition} : (A \to \neg B) \to (B \to \neg A) \\
  \text{Suffixing} : (A \to B) \to ((B \to C) \to (A \to C)) \\
  \text{Prefixing} : (A \to B) \to ((C \to A) \to (C \to B)) \\
  \text{Contraction} : (A \to (A \to B)) \to (A \to B)
  \]
Contraposition, suffixing, and prefixing are already valid in B in closely related forms:

- **Contraposition**: \((A \rightarrow \neg B) \models (B \rightarrow \neg A)\)
- **Suffixing**: \((A \rightarrow B) \models ((B \rightarrow C) \rightarrow (A \rightarrow C))\)
- **Prefixing**: \((A \rightarrow B) \models ((C \rightarrow A) \rightarrow (C \rightarrow B))\)
There are some things that are classically valid but not valid in R, namely:

\[ A \rightarrow (A \rightarrow A) \]
\[ (A \rightarrow B) \lor (B \rightarrow A) \]
\[ A \rightarrow (\neg A \rightarrow B) \]
\[ A \rightarrow (B \rightarrow B) \]
\[ A \rightarrow (B \rightarrow A) \]
\[ A \rightarrow (B \lor \neg B) \]
\[ (A \land \neg A) \rightarrow B \]
R Conditionals

- The conditional in R satisfies contraposition and transitivity.
- It is thus by itself not a very good candidate for counterfactuals, *ceteris paribus* principles, and other conditionals for which these inferences seem to fail.
Selection Functions

- Since R approximates classical logic, and its conditional satisfies various objectionable inferences, we might reasonably want to add a counterfactual conditional to R or weaker relevance logics.
- Let’s therefore add accessibility relations $R_A$ or selection functions $f_A$ for each formula $A$. 
As before, we add a conditional $\rightarrow$ with the truth conditions

$$v_w(A \rightarrow B) = 1 \iff f_A(w) \subseteq [B]$$

Validity in this logic, CB, is truth preservation at all normal worlds.
Tableaux for CB use the rules

\[ A > B, +x \]
\[ xr_A y \]
\[ \downarrow \]
\[ B, +y \]
Tableaux

\[ A \succ B, \neg x \]

\[ \downarrow \]

\[ xr_A j \]

\[ B, \neg j \]

\[ (j \ new) \]
We can extend this to CB+ in analogy with C+ by imposing the expected conditions on normal worlds.

- \( w \in N \Rightarrow f_w[A] \subseteq [A] \)
- \( w \in N \Rightarrow (w \in [A] \Rightarrow w \in f_w[A]) \)
Tableau Rules

\[ A, -0 \quad A, +0 \quad 0r_A 0 \]

\[ A > B, -0 \]
\[ \quad | \]
\[ \quad 0r_A j \]
\[ \quad A, +j \]
\[ \quad B, -j \]
\[ \quad j \text{ new} \]
CB+

- All C+ interpretations are CB+ interpretations (in which all worlds are normal), so CB+ is a sublogic of C+.
- So, the $>$ conditional in CB+ does not validate strengthening of the antecedent, transitivity, or contraposition.
- In fact, CB+ is a relevant sublogic of C+, for, if we let $W = N$ and $f_A(w) = [A]$, $>$ acts just like $\rightarrow$ in K*.
- Since irrelevant inferences fail in K*, they also fail in CB+.