Intuitionistic Logic

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So far we have been treating classical logic and extending it with additional, intensional connectives. This requires us to rearrange the semantics, giving us resources to account for possibility, necessity, counterfactuals, obligation, permission, etc. If $A$ is a classical tautology, then, in every system we’ve seen so far, $A$ is still valid. We have disputed the meaning of if...then, but we did it by adding a new connective to the system.

There are other logics, however, that challenge classical logic more fundamentally.

1 Logical Revisionism

Why might we want to reject classical logic?

Consider a mathematical question: Are there irrational numbers $a, b$ such that $a^b$ is rational? Consider $\sqrt{2}^{\sqrt{2}}$. It is either rational or irrational. If it’s rational, we’re done. If not, raise it to the $\sqrt{2}$ power. Then,

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^2 = 2$$

So, in either case, there are irrationals $a, b$ such that $a^b$ is rational.

This proof might worry someone who doesn’t think of doing mathematics as looking, as it were through a mental telescope, and discovering what is true in some abstract realm. Constructivists– those who think that we construct mathematical objects through mathematical activity– are likely to find the proof unacceptable. They think that, to justify an existential claim, you should construct the object satisfying the claim. The existence of objects is not independent of our constructive activity. The above proof doesn’t tell us which numbers $a$ and $b$ are. So, from a constructive point of view, we aren’t yet justified in asserting their existence.

There are two motivations for logical revisionism:

First, there is the epistemic conception of truth. This might follow crudely from a coherence theory of truth. Or it might be Peircean, depending on a notion of idealized circumstances, observers, etc. But the key idea is that, for something to be true, it must be possible for us to know it.

Michael Dummett has presented a general argument for revision on the basis of a conception of linguistic understanding. To understand a language, you must understand truth conditions for sentences of the language. For any sentence $S$ that you understand,
you must be able to divide the world into circumstances that would make \( S \) true and those that would make \( S \) false. If truth is not epistemic, it might be elusive; there might be truths such that no amount of investigation we could ever do would put us in a position to know whether they are true.

Consider in the decimal representation of \( \pi \), for each \( n \), somewhere there occurs in \( n \)-fold representation of \( n \). No matter how much investigation we do, we’re never in a position to render even a preliminary judgment about this. There are seven consecutive sevens around the three-millionth digit in the decimal expansion of \( \pi \) (rendering the classic example obsolete); but there is no way to be in a position to say anything about the general claim. So, this would be an elusive truth. Why is this a problem? We are trained in the use of our language. We learn meanings by way of use. Elusive sentences can’t be mastered, for we can’t use them appropriately or inappropriately; there are no circumstances of either kind. We must talk about truth in evidential terms. Truth must be connected to evidence in a substantive way.

When I assert that \( p \), I give \( p \) some positive epistemic status. What does that imply about disjunctions? Classically, we say that \( A \lor B \) is true iff \( A \) is true or \( B \) is true. But from an epistemic point of view this becomes: \( A \lor B \) has a positive epistemic status iff \( A \) does or \( B \) does. In the above proof, however, we tried to give \( A \lor B \) a positive status without doing so for either disjunct.

From this perspective, the law of excluded middle looks unacceptable. \( A \lor \neg A \) should not have a positive epistemic status unless \( A \) does or \( \neg A \) does. But we might not know anything about \( A \) or \( \neg A \). Note: \( \neg A \) has a positive epistemic status iff \( A \) has a negative epistemic status. And that will be true if we have a disproof of \( A \). Excluded middle thus says that we have a proof of \( A \) or a disproof of \( A \), or, more generally, that we have evidence for \( A \) or evidence against \( A \). But we might have no evidence relevant to \( A \) at all.

We will now need to distinguish \( A \) from \( \neg \neg A \). The latter means that attempts to refute \( A \) are bound to fail. But that isn’t itself evidence for \( A \).

Second, there is conventionalism or constructivism about the world. Suppose we project a structure onto the world. Then there is no reason why the law of excluded middle ought to be satisfied. \( A \) might be true by virtue of certain conventions; similarly for \( \neg A \). But our conventions might not specify either one. Consider John is courageous. We might think of that as saying something like If John is in danger, he won’t panic. We might think that he possesses this trait or he doesn’t. Think of the trait as a real feature of John, and this will seem natural. But this realism might seem unattractive. There are no independent traits out there somewhere; there is only the conditional. If John never faces danger, we might say we don’t know whether John is courageous. But we might also think that it is neither true nor false. There is nothing to ground its truth or its falsehood. John is not courageous, on a Stalnaker view of conditionals, will amount to If John is placed in danger, he would panic. But there is no evidence for that either, if John never faces danger. So, we are in no position to assert John is courageous or John is not courageous. This too motivates a revision of logic to get rid of the law of excluded middle.

This might motivate adding a third truth value. We’ll come back to that strategy soon. For now, however, it’s important to see that doing so would not be sufficient. In a
many-valued logic, if \( A \) is indeterminate, \( \neg A \) is too. And, if \( A \) and \( B \) are indeterminate, so are \( A \lor B \) and \( A \land B \). This succeeds in getting rid of the law of excluded middle. \( A \lor \neg A \) is indeterminate if \( A \) is. But the same will be true of \( A \land \neg A \). I can never, however, be in a position to prove both \( A \) and \( \neg A \)! There are other problems as well; \( A \) and \( \neg \neg A \) will end up equivalent. So, the above considerations do not argue for a many-valued logic.

2 Intuitionism

Intuitionism is an approach to the philosophy of mathematics and to mathematical practice pioneered by L. E. J. Brouwer. The central thesis of intuitionism is that mathematics has as its subject matter mental mathematical constructions. If they were all finite, then classical reasoning would be adequate to their description and analysis. But infinite structures require a new style of mathematical reasoning, and, correspondingly, a new logic.

Arend Heyting developed the reasoning used in intuitionistic mathematics into a subsystem of classical logic known as intuitionistic logic. The central idea is that truth is to be identified with provability.

There are two main justifications for intuitionistic logic. The first is metaphysical: the objects are to be viewed as constructed gradually over different stages of construction. The second is semantic: truth conditions are not generally epistemically accessible; we should replace talk of truth with talk of assertability, verifiability, or, in mathematics, provability. Dummett has argued powerfully that only the second can really serve as justification of revising the logic of the connectives. It’s not obvious why thinking of the domain of objects as constructed in either the collective or distributive senses would lead one to reinterpret the meaning of ‘not’, ‘if’, and ‘all’.

Here’s one way in which we might think about the meanings of the connectives once we replace talk to truth with talk of proof:

To prove \( A \land B \), prove both \( A \) and \( B \).

To prove \( A \lor B \), prove either \( A \) or \( B \).

To prove \( \rightarrow A \), prove that there is no proof of \( A \).

To prove \( A \not\rightarrow B \), prove that any proof of \( A \) can be turned into a proof of \( B \).

This fails to validate some classical principles. For example, the law of excluded middle, \( A \lor \neg A \) is not valid, for we may have neither a proof of \( A \) nor a proof that there is no proof of \( A \). Similarly, the law of double negation fails: a proof that there is no proof that there is no proof of \( A \) is not itself a proof of \( A \).

Kripke, in 1965, devised a possible-worlds semantics that encapsulates these ideas. Think of worlds as epistemic states. Accessibility amounts to potentiality; the accessible worlds are those I might end up in from my current epistemic state by way of further investigation. I will want reflexivity: my current epistemic state is clearly one I might end up in, since I’m in it. I will also want transitivity. If I am in epistemic state \( a \), and could end up in \( b \); and, if I were in \( b \), I might end up in \( c \); then I might end up in \( c \) (by way of \( b \)). This determines the structure of S4 (\( K_\rho \)).
We impose an additional hereditary condition: If $wRw'$ and $v(w, p) = 1$, then $v(w', p) = 1$. Sentence letters, once true, say true under further investigation. That is not necessarily true of other formulas. This is philosophically substantive, and controversial; it rules out fallibilism. There must be some class of statements that are indefeasible. If we come to know them, then that knowledge is immune from further undermining or revision.

Let an intuitionistic interpretation be a structure $< W, R, ν >$ for $K_\nu$ obeying the heredity condition: for every $p$ and $w \in W$, $v_w(p) = 1$ and $wRw' \Rightarrow v_{w'}(p) = 1$.

The assertability conditions are:

$$
v_w(A \land B) = 1 \Longleftrightarrow v_w(A) = v_w(B) = 1$$
$$v_w(A \lor B) = 1 \Longleftrightarrow v_w(A) = 1 \lor v_w(B) = 1$$
$$v_w(\neg A) = 1 \Longleftrightarrow \forall w'(wRw' \Rightarrow v_{w'}(A) = 0)$$
$$v_w(A \equiv B) = 1 \Longleftrightarrow \forall w'(wRw' \Rightarrow (v_w(A) = 0 \lor v_w(B) = 1))$$

We can reframe these elegantly, thinking of worlds as making sentences true:

$$w \models A \land B \iff w \models A \text{ and } w \models B$$
$$w \models A \lor B \iff w \models A \text{ or } w \models B$$
$$w \models \neg A \iff \forall w'(wRw' \Rightarrow w' \models \neg A)$$
$$w \models A \equiv B \iff \forall w'(wRw' \Rightarrow (w \models A \text{ or } w \models B))$$

We now have models in which excluded middle fails. Suppose we are at an epistemic state $w$ in which $p$ is false, but that subsequent investigation will show $p$ to be true. Then both $p$ and $\neg p$ fail, so $p \lor \neg p$ fails as well.

We can also easily get a model in which double negation elimination fails. Consider $w$, where $p, \neg p$ are false, and $w'$, where $p$ is true and $\neg p$ false. $\neg\neg p$ is true at $w$, since $\neg p$ is true at no accessible world. But then we have a world in which $\neg\neg p$ is true but $p$ is false.

Note that $p \models \neg\neg p$. To construct a countermodel, we would have to have worlds $w, w'$ such that $w \models p$ and $w' \models \neg p$, with $wRw'$. But then $w'$ would have to make $p$ false, which violates the heredity condition.

We can generalize this to formulas $A$. We can prove a heredity condition for formulas in general. If $wRw'$, then $w \models A \Rightarrow w' \models A$. So, imposing heredity on atomic formulas is enough to impose it on formulas in general.

Claim: If $w \models X$ and $wRw'$, $w' \models X$. Proof: By induction on complexity of formulas. Base: Sentence letters: trivial— that is the heredity condition. Inductive step: let $X = A \land B$. Assume theorem for all formulas of complexity less than that of $X$. If $w \models A \land B$, $w \models A$ and $w \models B$. But the theorem applies to $A$ and $B$, so $w' \models A, B, A \land B$. Disjunction is similar. Negation: Say $w \models \neg A$. Then $A$ is false at all worlds accessible from $w$. So, if $wRw'$, $w' \models \neg A$, since, by transitivity, anything accessible to $w'$ is also accessible to $w$. (NB: This argument doesn’t use the inductive hypothesis!) Conditional: Let $A \equiv B$ be true at $w$. At all further worlds, $A$ is false or $B$ is true. So, again, by transitivity, that will have to be true at each accessible world, including $w'$. (Again, we did not use inductive hypothesis.)
We would get classical logic out of this if we couldn’t have both \( p \) and \( \neg p \) false, or if we could get genealogy (the converse of heredity).

This combination makes sense. If I have a proof of \( p \), I have a disproof of any disproof of \( p \). But being able to disprove the possibility of a disproof of \( p \) isn’t itself a proof of \( p \).

Notice that the clauses for conjunction and disjunction (and, in quantification theory, \( \exists \)) are classical. The fragment of intuitionistic logic consisting just of \( \land \) and \( \lor \) is identical to a similar fragment of classical logic.

Negations collapse in pairs, so long as there are negations left. \( \neg \neg \neg A \equiv A \).

That suggests that there are three statuses: \( A \), \( \neg A \), \( \neg \neg A \). But \( A \lor \neg A \lor \neg \neg A \) is not valid. We can have \( wRw' \), \( wRw'' \) such that \( w \) gives all three falsehood, \( w' \) makes \( A \) true, and \( w'' \) makes \( A \) false. To understand this, think about what they mean. I may not be able to prove \( A \), or disprove \( A \), or disprove the possibility of disproving \( A \). Nevertheless, the intuitionist is willing to assert \( \neg \neg (A \lor \neg A \lor \neg \neg A) \).

It is easy, given these truth conditions, to translate intuitionistic logic into classical modal logic. (In fact, this is undoubtedly what inspired Kripke’s semantics; Gödel had thought of the translation in the early 1930s.) We can translate \( A \) as \( \Box \neg \neg A \) (for \( A \) has to fail in every accessible world) and \( A \vdash B \) as \( \Box (A \supset B) \) (for every accessible world must be such that \( A \) fails or \( B \) is true). The heredity condition allows us to translate all atomic formulas \( p \) as \( \Box p \). Let \( \pi \) be this translation function; it is easy to show that \( A \) is intuitionistically valid if \( A^\pi \) is valid in \( S4 \). We have a reflexive and transitive accessibility relation in both intuitionistic logic and \( S4 \). In addition, \( A \) is classically valid iff \( A^\pi \) is valid in \( S5 \).

Thus, the classical logician, acting as jungle linguist in the land of the intuitionists, can interpret them as having a lot of silent modals.

How does this link to the concept of proof? The worlds of an intuitionistic interpretation are in effect information states. \( wRw' \) iff \( w' \) is a possible extension of \( w \) by way of further proofs (or, more generally, through the acquisition of additional information). Clearly \( R \) is reflexive and transitive, and obeys heredity, since we think of information as being added in progressive stages.

An intuitionistic interpretation with just one world is a classical interpretation. So, anything true in all intuitionistic interpretations is true in all classical interpretations. Anything intuitionistically valid is thus also classically valid. The reverse is not true; excluded middle and double negation are counterexamples.

Also, if \( A \) is classically valid, \( \rightarrow A \) is intuitionistically valid. Every classical validity has an intuitionistically valid counterpart. Think about a finite Kripke structure, and consider the worlds at the ends of accessibility paths. These worlds are utterly classical, for the only world accessible to each is itself. The modal logic in these worlds, in other words, is the trivial system in which \( A \equiv \Box A \). The only world I need to look at is the world I am in, at the end of the Kripke structure. So, given the hereditary character of intuitionistic truth, we can never have something coming out true that would fail in a classical world.

Say I assert \( \rightarrow A \) at \( w \). Under a modal translation, this is \( \Box \Diamond A \). It must be the case that \( A \) is true at some accessible world.

What if there aren’t any end worlds? Then some fancy footwork is necessary. A key idea in the background is that proofs are finite and noncircular; whenever there is
a Kripke model for a set of formulas, there is a finite Kripke model of that set without loops.

The intuitionistic jungle linguist can now interpret the classical logician: classical natives have lots of silent double negations.

This is true of sentential logic, but not of quantification theory. \((Fx \vee \neg Fx)\) is not even intuitionistically valid.

Proof systems: Usually, one can obtain intuitionistic logic simply by dropping the rule of double negation. This is not available in tableaux. In the tableau system, you must start tracking truth and falsity of claims independently. So, I must write \(p, +0\) or \(p, −0\) to say whether \(p\) gets T or F at world 0.