

1 Normal Semantics

Call the logic we have developed so far $K$ (for Kripke). $K$ is the most general normal modal logic. We noted some inferences that are valid in $K$. For example, where $\vdash_K$ indicates entailment in $K$:

- If $X \vdash_K A$, then $\Box X \vdash_K \Box A$ (where $\Box X = \{ B : B \in X \}$)
- If $A \vdash_K B$, then $\Diamond A \vdash_K \Diamond B$.
- If $X, B \vdash_K A$, then $\Box X, \Diamond B \vdash_K \Diamond A$.

Two things that follow from these generalizations are especially useful, and can be taken as axiomatizing $K$:

- Distribution: $\Box A \land \Box (A \supset B) \supset \Box B$
- Necessitation: $\vdash_K A \Rightarrow \vdash_K \Box A$

We derive other normal logics from $K$ by imposing constraints on the accessibility relation. Where $c$ is such a constraint, a $c$-interpretation is an interpretation obeying $c$. We can write $X \vdash_c A$ to mean that $A$ holds on every $c$-interpretation making every member of $X$ true. Because every normal modal logic defines validity on some subset of the interpretations available in $K$, every such logic extends $K$ in the sense that anything valid in $K$ is valid in any normal modal logic.

Here are some of the most important constraints:

- Reflexivity ($\rho$): $\forall w R w$
- Symmetry ($\sigma$): $\forall w_1 w_2 (w_1 R w_2 \supset w_2 R w_1)$
- Transitivity ($\tau$): $\forall w_1 w_2 w_3 ((w_1 R w_2 \land w_2 R w_3) \supset w_1 R w_3)$
- Euclidean property ($\epsilon$): $\forall w_1 w_2 w_3 ((w_1 R w_2 \land w_1 R w_3) \supset w_2 R w_3)$
Extendability ($\eta$; also known as Seriality): $\forall w_1 \exists w_2. w_1 R w_2$

If $R$ is reflexive, it is extendable. If $R$ is extendable, symmetric, and transitive, then it is reflexive. Imposing these constraints on $K$ yields the following logics:

$$T = K_{\rho}$$

This system, discovered by Feys (as $T$) in 1937 and independently by von Wright (as $M$) in 1951, has infinitely many modalities. Necessity differs from possible necessity, necessary necessity, possibly necessary necessity, possibly possible necessity, necessarily necessary necessity, necessarily possible necessity, etc. Every world here is accessible to itself. Reflexivity guarantees the validity of $T$’s characteristic axiom, $\models_T \square A \rightarrow A$. (Whatever is necessary is true.) This seems essential for most conceptions of necessity, though it of course fails for deontic modalities, belief, etc. In speaking of this as the system’s characteristic axiom, I mean that adding it to the axioms for $K$ yields a sound and complete axiomatization of $T$. $T$ differs from the Lewis systems $S1$, $S2$, and $S3$ in asserting the necessity of logical truths: $\models_T A \Rightarrow \models_T \square A$.

Tableau rule:

$$\text{iri}$$

This system, investigated by von Wright, has become known as standard deontic logic, the logic of obligation and permission. (Replace $\square$ with $O$, and $\Diamond$ with $P$.) We can think of the set of accessible worlds as ideal worlds; everything obligatory in a world is true in all its ideals. Extendability guarantees that every world has at least one ideal, a world in which all its obligations are fulfilled. That means that the obligations in any world are consistent and thus capable of being fulfilled together in some world. For that reason, standard deontic logic has become controversial; it seems to rule out conflicting obligations. The characteristic axiom: $\models_D \square A \supset \Diamond A$. (What is necessary is possible, or, in the deontic case, what is obligatory is permissible.) $D$ also has infinitely many modalities. Tableau rule:

$$\text{irj}$$

($j$ new)

$$B = K_{\rho\sigma}$$

This is the Brouwersche system, so-called because of a rather vague similarity to intuitionistic logic, developed by Kripke in 1963. It has the characteristic axiom $\models_B A \supset \square \Diamond A$. (Whatever is true is necessarily possible.) It has infinitely many modalities. Tableau rule for $\sigma$:
S4 = $K_{\rho\tau}$

S4 is one of the original Lewis-Langford systems. It is a popular candidate for representing nomological and other senses of necessity. It is in effect a theory of partial orders. It has 14 modalities: $A$, $\Box A$, $\Diamond A$, $\Box \Diamond A$, $\Box \Box A$, $\Diamond \Box A$, $\Box \Diamond \Box A$, and their negations. Its characteristic axiom: $\Box A \supset \Box \Box A$. Tableau rule for $\tau$:

\[
\begin{array}{c}
irj \\
jri
\end{array}
\]

S5 = $K_{\rho\nu \varepsilon} = K_{\rho\varepsilon}$

S5, the strongest of the original Lewis-Langford systems, results from taking the accessibility relation as an equivalence relation. It also results from taking it to be a universal relation, which has the effect of making accessibility irrelevant. S5 thus represents the Leibnizian view of necessity as truth in all possible worlds. It distinguishes 6 irreducible modalities: $A$, $\Box A$, $\Diamond A$, and their negations. Its characteristic axiom, $\Diamond A \supset \Box \Diamond A$, has the effect of making all modalities but the last in a string redundant. Thus, $\Box \Diamond \Box \Diamond A$ is equivalent to $\Diamond A$. That axiom holds iff the accessibility relation is Euclidean. Tableau rule for $\varepsilon$ (usually not used for S5, since the rules for $\rho$, $\sigma$, and $\tau$ suffice):

\[
\begin{array}{c}
irj \\
jrk \\
irk
\end{array}
\]