Contemporary students of logic tend to think of the logic of the connectives as the most basic area of the subject, which can then be extended with a logic of quantifiers. Historically, however, the logic of the quantifiers, in the form of the Aristotelian theory of the syllogism, came first. Truth conditions for negation, conjunction, and disjunction were well understood in ancient times, though not until Leibniz did anyone appreciate the algebraic features of these connectives. Approaches to the conditional, meanwhile, depended on drawing an analogy between conditionals and universal affirmative propositions. That remained true throughout the ancient, medieval, and early modern periods, and extended well into the nineteenth century, when Boole constructed an algebraic theory designed to handle sentential and quantificational phenomena in one go. The strength of the analogy, moreover, undercuts a common and otherwise appealing picture of the history of logic, according to which sentential and quantificational threads developed largely independently and, sometimes, in opposition to each other, until Frege wove them together in what we now consider classical logic. Frege did contribute greatly to our understanding of the connectives as well as the quantifiers. But his contribution consists in something other than unifying them into a single theory.

1 Aristotelian Foundations

Aristotle (384–322 BC), the founder of logic, develops a logic of terms rather than a logic of propositions. He nevertheless argues for and against certain broad metaprin-ciples with direct relevance to propositional logic. In Metaphysics Γ 4, for example, he famously argues in favor of the principle of noncontradiction in the form “it is impossible for anything at the same time to be and not to be,” maintaining that anyone who opposes it is “no better than a vegetable.” In De Interpretatione 9 he argues against the principle of bivalence, contending that, if every proposition is either true or false, everything must be taken to happen by necessity and not by chance. He correspondingly denies that “There will be a sea battle tomorrow” is either true or false. These are important discussions, raising central logical and philosophical issues. But they fall far short of a logic of propositional connectives.

That has not stopped scholars from finding the core of such a logic in Aristotle. Lear (1980), for example, finds in Prior Analytics I, 23’s discussion of the reduction of syllogisms to first figure an account of indirect proof, which seems fair enough; Aristotle does rely on the pattern $A, B \vdash C \equiv A, \lnot C \vdash \lnot B$. Slater (1979) notes the par-
allel between Aristotle’s logic of terms and a Boolean propositional logic, constructing an Aristotelian propositional logic by bending the latter to the former rather than the reverse. It is not hard to do; since Aristotle takes universals as having existential import, and Boole takes universals and conditionals as analogues, interpret a conditional \( A \to B \) as holding when \( \emptyset \subset [A] \subseteq [B] \), where \([A]\) and \([B]\) are sets of cases in which \( A \) and \( B \), respectively, are true. This has the important implication that no true conditional has a necessarily false antecedent. Still, to obtain a logic from this, one needs to take conjunction as corresponding to intersection, disjunction as corresponding to union, and negation as corresponding to complement—none of which is particularly well-motivated within Aristotle’s logic as such.

One seemingly propositional argument Aristotle makes is today almost universally considered fallacious. In *Prior Analytics* II, 4 he argues:

But it is impossible that the same thing should be necessitated by the being and by the not-being of the same thing. I mean, for example, that it is impossible that \( B \) should necessarily be great if \( A \) is white and that \( B \) should necessarily be great if \( A \) is not white. For whenever if this, \( A \), is white it is necessary that, \( B \), should be great, and if \( B \) is great that \( C \) should not be white, then it is necessary if \( A \) is white that \( C \) should not be white. And whenever it is necessary, if one of two things is, that the other should be, it is necessary, if the latter is not, that the former should not be. If then if \( B \) is not great \( A \) cannot be white. But if, if \( A \) is not white, it is necessary that \( B \) should be great, it necessarily results that if \( B \) is not great, \( B \) itself is great. But this is impossible. For if \( B \) is not great, \( A \) will necessarily not be white. If then if this is not white \( B \) must be great, it results that if \( B \) is not great, it is great, just as if it were proved through three terms.

\( (57a36–57b17) \)

It is not clear what relation Aristotle has in mind by “necessitated,” but let’s provisionally represent it symbolically with an arrow. Then Aristotle appears to be arguing for the thesis that \( \neg((A \to B) \land (\neg A \to B)) \). In keeping with this propositional schema, let’s take \( A \) and \( B \) as standing for propositions rather than objects, as in Aristotle’s text. The argument seems to go as follows:

1. \( A \to B \) (assumption)
2. \( \neg A \to B \) (assumption)
3. \( A \to B, B \to \neg C \Rightarrow A \to \neg C \) (transitivity)
4. \( A \to B \Rightarrow \neg B \to \neg A \) (contraposition)
5. \( \neg B \to \neg A \) (modus ponens, 1, 4)
6. \( \neg B \to B \) (transitivity, 5, 2)
7. \( \neg(\neg B \to B) \) (??)
Aristotle gives no explanation for why a proposition cannot be necessitated by its own negation. Perhaps he has in mind the central idea of connexive implication, that the antecedent of a true conditional must be compatible with the conclusion (McCall 1966). But perhaps he simply thinks, given the parallel with universal propositions and their existential import, that antecedents of true conditionals must be possible. Given a necessitation account of the conditional, these are of course equivalent. But they are distinct on other, weaker accounts.

Leave that aspect of the argument aside for a moment. Aristotle does use several interesting principles in this argument, which we might formalize as follows:

- Transitivity of the conditional: \( A \rightarrow B, B \rightarrow C \Rightarrow A \rightarrow C \)
- Contraposition of the conditional: \( A \rightarrow B \Rightarrow \neg B \rightarrow \neg A \)
- Modus ponens on the conditional: \( A \rightarrow B, A \Rightarrow B \)

His use of modus ponens in this passage, however, is metatheoretic. In general, Aristotle’s text is ambiguous between a conditional reading and an entailment reading. It would be justifiable to interpret the rules as

- Transitivity of entailment: \( A \models B, B \models C \Rightarrow A \models C \)
- Contraposition of entailment: \( A \models B \Rightarrow \neg B \models \neg A \)
- Modus ponens on entailment: \( (A \models B, A) \Rightarrow B \)

This kind of ambiguity presents a serious problem to anyone seeking to find a propositional logic implicit in Aristotle. It also, as we shall see, presents problems to anyone seeking to interpret medieval discussions of consequences.

## 2 Stoic Logic

The first explicit theory of propositional connectives was developed by a collection of thinkers known as the Stoics. They took logic seriously. Diogenes Laertius reports that when Diodorus Cronos could not solve a logical puzzle the King posed to him at a banquet, he died, heart-broken (Vitae II, 111). Philetas of Cos, in the epitaph he asked to be placed on his tombstone, blamed the Liar paradox for his death (Athen. IX, 401C).

The Stoic definition of argument is strikingly modern. “An argument is a system consisting of premises and a conclusion. Those propositions which are agreed upon for the establishment of the conclusion are called ‘premises,’ and the proposition which is established from the premises is called the ‘conclusion,’ as, for instance, in the following argument:

If it is day, then it is light.
It is day.
Therefore, it is light.
The proposition ‘It is light’ is the conclusion and the others are premises” (Sextus, Hyp. Pyrrh., II, 135ff., in Mates 1961, 110).

The Stoics also offer a general definition of validity: “the criterion for validity is that an argument is valid whenever the conclusion follows logically from the conjunction of the premises” (Sextus, Adv. Math., VIII, 415, in Mates 1961, 106). “Of arguments, some are conclusive and some are inconclusive. Inconclusive arguments are those which are such that the denial of the conclusion is compatible with the conjunction of the premises:

If it is day, then it is light.
It is day.
Therefore, Dion is walking.” (Diogenes Laertius, Vitae VII, 78, in Mates 1961, 114)

This is essentially Aristotle’s conception of validity. Like Aristotle, the Stoics sometimes load it down with extraneous considerations. Diogenes Laertius, for example, reports the Stoics as having defined arguments as having two premises (Vitae VII, 76, in Mates 1961, 114). Kneale and Kneale take this as accurate (162–163) on grounds that all the basic Stoic inference patterns have two premises.

The basic unit of Stoic logic is not the term, as in Aristotle, but the proposition (lecton). A sentence signifies a proposition, a “sayable” that is a bearer of truth value. Sentences are tangible, things spoken or written in a particular language; propositions are intangible and independent of any particular language. They are abstract contents of sentences. ‘It’s raining,’ ‘Piove,’ ‘Es regnet,’ ‘Il pleut,’ ‘Det regner,’ ‘Esta lloviendo,’ and ‘Wa mvua’ are different sentences in different languages, but, if said of the same place and time, have the same content, signifying the same lecton. Not all sayables are propositions. Among the complete sayables are commands and questions; among the incomplete are predicates. Incomplete sayables turn into complete sayables when appended to names. Propositions, according to the Stoics, may change in truth value. ‘It is raining’ may be false now but true tomorrow.

Propositions, then, are expressible contents that have truth values. The Stoics have a strong commitment to bivalence; every proposition is either true or false. There are no truth value gaps. They are similarly committed to noncontradiction; there are no truth value gluts. They distinguish between these metatheoretic claims and their expression in the law of excluded middle (A or not A) and noncontradiction (not both A and not A).

2.1 The Nature of the Conditional

Stoic logic is best known for the controversy over the nature of the conditional. “Even the crows on the rooftops caw about which conditionals are true,” Callimachus writes (Sextus, Adv. Math., I, 309–310). A conditional, Chrysippus stipulates, is a compound proposition formed with the connective ‘if’ (ei, eiper; Sextus Adv. Math. VIII, 109). Under what conditions is such a sentence true? When “the second part follows from the first,” as Sextus says (Adv. Math. VIII, 111); but under what conditions does the consequent follow from the antecedent?

Sextus outlines four competing answers:
For Philo says that a true conditional is one which does not have a true antecedent and a false consequent; e.g., when it is day and I am conversing, “If it is day, then I am conversing”;

but Diodorus defines it as one which neither is nor ever was capable of having a true antecedent and a false consequent. According to him, the conditional just mentioned seems to be false, since when it is day and I have become silent, it will have a true antecedent and a false consequent; but the following conditional seems true: “If atomic elements of things do not exist, then atomic elements of things do exist,” since it will always have the false antecedent, “Atomic elements of things do not exist,” and the true consequent, “Atomic elements of things do exist.”

And those who introduce “connection” or “coherence” say that a conditional holds whenever the denial of its consequent is incompatible with its antecedent; so that, according to them, the above-mentioned conditionals do not hold, but the following is true: “If it is day, then it is day.”

And those who judge by “suggestion” declare that a conditional is true if its consequent is in effect included in its antecedent. According to these, “If it is day, then it is day,” and every repeated conditional will probably be false, for it is impossible for a thing itself to be included in itself. (HP II, 110; cf. Adv. Math. VIII, 112ff.)

It is hard to understand the final option; what is inclusion? And, if it is impossible for a thing to be included in itself, why are repeated conditionals “probably” rather than necessarily false?¹ The other three, however, seem straightforward:

- **Philo**: $A \rightarrow B \iff \neg(A \land \neg B)$
- **Diodorus**: $A \rightarrow B \iff$ it is always the case that $\neg(A \land \neg B)$
- **Chrysippus**: $A \rightarrow B \iff$ it is necessary that $\neg(A \land \neg B)$

Philo analyzes conditionals as material conditionals in the modern sense: “a conditional holds unless its antecedent is true and its consequent is false” (Sextus, Adv. Math. VIII, 332); “… a true conditional is one which does not have a true antecedent and a false consequent” (HP II, 104). Sextus outlines the Philonian position in terms of a truth table:

Philo said that the conditional is true whenever it is not the case that its antecedent is true and its consequent false; so that, according to him, the conditional is true in three cases and false in one case. For it is true when the antecedent is true and the consequent is true. For example, “If it is day, it is light.” Again, it is true when the antecedent is false and the consequent is false. For example, “If the earth flies, then the earth has wings.” It is

¹We do not mean to suggest that this option cannot be given interesting interpretations. O’Toole and Jennings 2004, for example, develop a connexivist account. Bonevac, Dever, and Sosa (forthcoming) develop a neighborhood semantics that drops idempotence. But we do not have enough evidence concerning this fourth option to know what its advocates had in mind.
also true whenever the antecedent is false and the consequent is true. For example, “If the earth flies, then the earth exists.” It is false only when the antecedent is true and the consequent is false, as, for example, “If it is day, then it is night.” (Adv. Math. VIII, 112ff.; cf. Hyp. Pyrrh. II, 104ff.)

We unfortunately have no record of debates between Philo and Stoics holding contrary positions. So, we do not know what arguments Philo and his followers used to support their position. Nor do we know what arguments Diodorus, Chrysippus, and their followers brought against it.

Chrysippus analyzes conditionals as strict conditionals. The sense of necessity he has in mind is not clear; some examples suggest logical necessity, but some, such as Diogenes’s “If it is day, it is light,” could be read as physical necessity (Mates 1961, 48). Diodorus takes an intermediate position, analyzing conditionals in temporal terms.

The Stoics link argument validity to conditionals in a way that, at first glance, appears familiar from the deduction theorem. In fact, however, its adequacy depends on the interpretation of the conditional. Recall the general definition of validity: “the criterion for validity is that an argument is valid whenever the conclusion follows logically from the conjunction of the premises” (Sextus, Adv. Math., VIII, 415, in Mates 1961, 106). Equivalently, an argument is valid when the truth of the premises is incompatible with the falsehood of the conclusion. Take the conditional with the conjunction of the premises as antecedent and the conclusion as consequent to be an argument’s associated conditional. Implication, as W. V. Quine stresses, is the validity of the conditional. So, an argument is valid if and only if its associated conditional is valid, that is, a logical truth. The Stoics, however, appear to have held that an argument is valid if and only if its associated conditional is true:

Some arguments are valid and some are not valid: valid, whenever the conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion, is true (Sextus, Hyp. Pyrrh. II, 135ff., in Mates 1961, 110).

So, then, an argument is really valid when, after we have conjoined the premises and formed the conditional having the conjunction of premises as antecedent and the conclusion as consequent, it is found that this conditional is true (Sextus, Adv. Math., VIII, 415, in Mates 1961, 107).

An argument is valid whenever there is a true conditional which has the conjunction of the premises as its antecedent and the conclusion as its consequent (Sextus, Adv. Math., VIII, 426, in Mates 1961, 108; see also Hyp. Pyrrh. II, 113, in Mates 1961, 110).

For a proof is held to be valid whenever its conclusion follows from the conjunction of its premises as a consequent follows from its antecedent, such as [for]:

If it is day, then it is light.
It is day.
Therefore, it is light.
These are all from Sextus, so it is possible that he confused logical truth with truth simpliciter. It is also possible that the Stoics did so. If there is no confusion, however, these passages support Chrysippus’s understanding of the conditional. The truth of its associated conditional, when that is interpreted materially, hardly suffices for validity. The truth of a strict conditional interpreted as indicating logical necessity, in contrast, does.

2.2 Stoic Theories of Conjunction, Disjunction, and Negation

Stoic theories of the relatively uncontroversial connectives—conjunction, disjunction, and negation—are surprisingly contemporary in feel. The Stoics characterize these connectives semantically, in terms of truth conditions, but also in terms of syntactic axioms or rules intended to capture their inferential behavior. They appear to have a concept of scope (kurieuei) that matches our contemporary notion as well as a clear distinction between atomic and molecular propositions (Sextus, Adv. Math. VIII, 89ff., 93, 108, in Mates 1961, 97; Diogenes Laertius, Vitae VII, 68, in Mates 1961, 112–113). The contemporary feel of Stoic doctrines has led many historians of Stoic logic (Lukasiewicz 1934, Mates 1961, Kneale and Kneale 1962, and Bochenski 1963, for example) to treat Stoic logic as a version of contemporary propositional logic, representing Stoic formulas in modern symbolism. As O'Toole and Jennings (2004) remind us, however, this can be dangerous. Stoic concepts often differ from our own. That said, the parallels between Stoic and twentieth-century theories are remarkable.

Negation. Let’s begin with what is in some ways the simplest propositional connective, negation. Stoic truth conditions for negation appear to be congruent with contemporary ones: “Among propositions, those are contradictories of one another, with respect to truth and falsehood, of which the one is the negation of the other. For example, ‘It is day’ and ‘It is not day’ (Diogenes Laertius, Vitae VII, 73, in Mates 1961, 113). The Stoics warn that we must use a negative particle that has scope over the entire proposition in order to obtain a negation. The truth condition they have in mind is evidently

\[ \neg A = 1 \iff [A] = 0. \]

They have a similarly modern conception of negation’s inferential role, advocating a rule of double negation: “A negative proposition is one like ‘It is not day.’ The double-negative proposition is a kind of negative. For a double negation is the negation of a negation. For example, ‘Not: it is not day.’ It asserts, ‘It is day’” (Diogenes Laertius Vitae VII, 70, in Mates 1961, 113). Similarly: “For ‘Not: not: it is day’ differs from ‘It is day’ only in manner of speech” (Alexander, In An. Pr., Wallies, 18). We may formulate the rule schematically as

\[ \neg \neg A \iff A. \]

No source lists this rule among the basic principles of Stoic logic; it must have been viewed as a derived rule.
Conjunction. Kneale and Kneale are dismissive of Stoic views on conjunction: “Of conjunction the Stoics had not much to say” (160). That is true only in the sense that contemporary logic has “not much to say” about conjunction; what the Stoics say might be viewed as a generalization of typical modern accounts.

The Stoic semantic characterization of conjunction is simple: “a conjunction holds when all the conjuncts are true, but is false when it has at least one false conjunct” (Sextus, *Adv. Math.* VIII, 125, in Mates 1961, 98); “in every conjunction, if one part is false, the whole is said to be false, even if the others are true” (Gellius, *Noctes Atticae* XVI, viii, 1ff; in Mates 1961, 122–123). Stoic conjunction is not binary, as contemporary conjunction is. It is multigrade, capable of linking two or more propositions together. In this respect it is closer to the English ‘and’ than the contemporary binary conjunction connective is. To reflect its truth conditions accurately, we must write

\[\land(A_1, \ldots, A_n) = 1 \iff \forall i \in \{1, \ldots, n\}[A_i] = 1.\]

Surprisingly, one does not find among Stoic rules anything corresponding to modern rules for conjunction:

\[A_1, \ldots, A_n \Rightarrow \land(A_1, \ldots, A_n)\]
\[\land(A_1, \ldots, A_n) \Rightarrow A_i.\]

Instead, we find conjunction appearing only within the scope of negation.

Disjunction. Stoic logic incorporates at least two conceptions of negation. One corresponds to a multigrade version of our familiar inclusive disjunction:

\[\lor(A_1, \ldots, A_n) = 1 \iff \exists i \in \{1, \ldots, n\}[A_i] = 1.\]

This, however, is not the primary disjunction connective. Galen, in fact, speaks of it as “pseudo-disjunction”:

“Therefore, in consideration of clarity together with conciseness of teaching, there is no reason not to call propositions containing complete incom- patibles ‘disjunctions,’ and those containing partial incompatibles ‘quasi-disjunctions.’ ... Also, in some propositions, it is possible not only for one part to hold, but several, or even all; but it is necessary for one part to hold. Some call such propositions ‘pseudo-disjunctions,’ since disjunctions, whether composed of two atomic propositions or of more, have just one true member.” (Galen, *Inst. Log.*, 11, 23ff., in Mates 1961, 118)

Usually, inclusive disjunction appears not as disjunction at all, but instead as a negated conjunction—indicating that Stoics understood what are often misleading called De Morgan’s laws. Philoponus refers to disjunction in this sense as quasi-disjunction, which he defines in terms of a negated conjunction; “It proceeds on the basis of propositions that are not contradictory” (Scholia to Ammonius, *In An Pr.*, Praefatio, xi, in Mates 131).

\[\text{Galenv restricts the truth of conjunctions to cases in which the conjuncts are neither consequences of one another nor incompatible, but he recognizes that others do not adopt this restriction (Inst. Log. 10, 133ff, in Mates 1961, 118; see also 32, 13ff., in Mates 1961, 120–121.}\]
The most concise statements of truth conditions for disjunctions in the primary sense appear in Gellius—“Of all the disjuncts, one ought to be true and the others false” (Noctes Atticae XVI, viii, 1ff., in Mates 1961, 123)—in Galen: “disjunctions have one member only true, whether composed of two simple propositions or more than two” (Inst. Log. 5.1)—and in Sextus: “a true disjunction announces that one of its disjuncts is true, but the other or others false” (Hyp. Pyrrh. 2.191). This suggests a truth condition requiring that exactly one disjunct be true:

$$\left[\oplus(A_1, \ldots, A_n)\right] = 1 \iff \exists i \in \{1, \ldots, n\}[A_i] = 1.$$

Kneale and Kneale (1962, 162), Bochenski (1963, 91), Mates (1961, 51), and Lukasiewicz (1967, 74) all take Stoic disjunction in the primary sense as exclusive disjunction. In the binary case, the two are equivalent. In general, however, they are not. As O’Toole and Jennings (2004, 502) observe, linking a sequence of n propositions with exclusive disjunctions yields something weaker than what the Stoics intend, something that is true if and only if an odd number of them are true. The Stoic concept, however, might reasonably be viewed as a multigrade generalization of exclusive disjunction.

Why did Stoic logicians take $\oplus$ rather than $\lor$ as primary? No answer in terms of natural language semantics seems plausible; Greek and Latin, like English, have no connective best understood as expressing $\oplus$ (McCawley 1980, O’Toole and Jennings 2004). One response is that $\lor$ is easily definable in terms of conjunction and negation:

$$\lor(A_1, \ldots, A_n) \iff \neg \land (\neg A_1, \ldots, \neg A_n)$$

That, indeed, is how the Stoics typically understand it. Defining disjunction in their primary sense, $\oplus$, in contrast, is considerably more complicated. Using inclusive disjunction:

$$\oplus(A_1, \ldots, A_n) \iff \lor((\land(A_1, \neg A_2, \ldots, \neg A_n), \land(\neg A_1, A_2, \ldots, \neg A_n), \ldots, \land(\neg A_1, \neg A_2, \ldots, A_n))$$

Using conjunction and negation alone:

$$\oplus(A_1, \ldots, A_n) \iff \neg \land (\neg \land(A_1, \neg A_2, \ldots, \neg A_n), \neg \land(\neg A_1, A_2, \ldots, \neg A_n), \ldots, \neg \land(\neg A_1, \neg A_2, \ldots, A_n))$$

Adding disjunction in what the Stoics considered its primary sense did not expand the expressive power of the language from a theoretical point of view, but it did make the system capable of expressing certain propositions much more economically.

There may be a third conception of disjunction in the Stoics, relating it to the conditional. Galen describes some Stoics as defining disjunctions in terms of conditionals: ‘A or B’ is equivalent to ‘If not A, then B’ (Inst. Log. 8, 12ff., in Mates 1961, 117–118). The account of disjunction that results, of course, depends on the analysis of the conditional. If the conditional is Philonian, the result is inclusive disjunction. If the conditional is Diodoran, it is inclusive disjunction prefixed with an “always” operator. If the conditional is Chrysippian, it is inclusive disjunction prefixed with a necessity operator. In no case does such a conception yield $\oplus$.

3There is some confusion among sources about the Stoic truth conditions for disjunction. Diogenes Laertius, for example, speaks only of falsehood, and seems to treat disjunction as binary: “This connective announces that one or the other of the propositions is false” (Vitae VII, 72, in Mates 1961, 113).
2.3 The Stoic Deduction System

The Stoics develop a deduction system for propositional logic, of which we have substantial fragments. They have a conception of completeness, and, as we have seen, have a formal semantics capable of giving real content to their claim that the system is complete (Diogenes Laertius, *Vita* VII, 78). Unfortunately, that claim is unfounded. It is not difficult, however, to supplement the Stoic rules to obtain a complete system.


1. If $A$ then $B$; $A$; therefore $B$ (modus ponens)
2. If $A$ then $B$; not $B$; therefore not-$A$ (modus tollens)
3. It is not the case that both $A$ and $B$; $A$; therefore not-$B$
4. Either $A$ or $B$; $A$; therefore not-$B$
5. Either $A$ or $B$; not $A$; therefore $B$

The fourth axiom indicates that the disjunction intended is not inclusive “pseudo-disjunction” but one implying incompatibility. Also, the use of ‘therefore’ rather than ‘if’ indicates that these are what we would now consider rules of inference rather than axioms. So, it seems fair to represent these symbolically as rules

1. $A \rightarrow B, A \vdash B$
2. $A \rightarrow B, \neg B \vdash \neg A$
3. $\neg (A \land B), A \vdash \neg B$
4. $A \oplus B, A \vdash \neg B$
5. $A \oplus B, \neg A \vdash B$

This representation immediately raises a question. Stoic conjunction and disjunction are multigrade, not binary. So, why do the axioms, as stated in all the available sources, treat them as binary? In the case of conjunction, there is no problem, for extended conjunctions are equivalent to conjunctions built up in binary fashion. That is not true for Stoic disjunction; $\oplus (A, B, C)$ is not equivalent to $A \oplus (B \oplus C)$. We may perhaps more accurately capture the intentions of the Stoics by writing multigrade rules, assuming that the binary statement was only a convenience, a shorthand for something harder to state in words. One look at the symbolic representations makes it clear why such a simplification would seem desirable. If we think of conjunction and disjunction as
applying to two or more propositions, in fact, we would need to state the third, fourth, and fifth axioms as above in addition to what appears below. We can forego that, however, if we assume that \( \land(A) = \oplus(A) = A \).

1. \( A \rightarrow B, A \vdash B \)
2. \( A \rightarrow B, \neg B \vdash \neg A \)
3. \( \neg \land(A_1, ..., A_n), A_i \vdash \neg \land(A_1, ..., A_{i-1}, ..., A_{i+1}, ..., A_n) \)
4. \( \oplus(A_1, ..., A_n), A_i \vdash \land(\neg A_1, ..., \neg A_{i-1}, \neg A_{i+1}, ..., \neg A_n) \)
5. \( \oplus(A_1, ..., A_n), \neg A_i \vdash \oplus(A_1, ..., A_{i-1}, A_{i+1}, ..., A_n) \)

**Rules (Themata).** The Stoics are widely reported to have used four rules (Galen, *De Hipp. et Plat. Plac.* ii 3 (92); Alexander, *In Ar. An. Pr. Lih. I Commentarium*, Wallies 284), of which we have only two (Kneale and Kneale 169):

1. \( A, B \vdash C \Rightarrow A, \neg C \vdash \neg B \) (and \( A, B \vdash C \Rightarrow B, \neg C \vdash \neg A \))
2. \( A, B \vdash C \) and \( \Gamma \vdash A \Rightarrow \Gamma, B \vdash C \)

Just as the Stoic axioms are not what we today consider axioms, but instead simple rules of inference, so these are complex rules of inference. They do not allow us to write formulas of certain shapes if we already have formulas of certain shapes; they instead allow us to infer from a derivation another derivation. In that respect they are more like indirect proof and conditional proof than modus ponens. Indeed, the first rule is a general form of indirect proof; it encodes Aristotle’s practice of using reductio proofs for certain syllogistic forms. The second is a version of cut.

What are the missing two? Historians of logic have made a variety of conjectures. Sources describe them as close to the cut rule. Bobzien (1996) reconstructs them as

- \( A, B \vdash C \) and \( A, C \vdash D \Rightarrow A, B \vdash D \) (and \( A, B \vdash C, B, C \vdash D \Rightarrow A, B \vdash D \))
- \( A, B \vdash C \) and \( \Gamma, A, C \vdash D \Rightarrow \Gamma, A, B \vdash D \) (and \( A, B \vdash C \) and \( \Gamma, B, C \vdash D \Rightarrow \Gamma, A, B \vdash D \))

If that is correct—and the hypothesis does explain why the sources found it impossible to keep three of the four rules straight—the Stoics could easily have simplified their system by putting cut more generally:

- \( \Gamma \vdash A \) and \( \Delta, A \vdash B \Rightarrow \Gamma \cup \Delta \vdash B \)

Is the Stoic system complete? They claimed it to be (Diogenes Laertius 7.79, Sextus, *Hyp. Pyrrh.* 2.156–157, 166–167, 194). Unlike Aristotle or even Frege, moreover, they had a well-defined semantics, even if the concept of validity itself remained imprecise. We have no record of any argument for completeness, however, much less a proof of it. We do have a record of some of the theorems that the Stoics derived. The most interesting, from the perspective of completeness, is double negation.\(^4\)

\(^4\)Other theorems include:
As it stands, it is obvious that the Stoic system is incomplete; there is nothing corresponding to a rule of conditional introduction. It is easy to show, for example, that \( p \rightarrow p \) is not provable (see Mueller 1979). This is not surprising, for it is precisely with respect to such a rule that the Stoic interpretations of the conditional vary. Philo, Diodorus, and Chrysippus can all agree on modus ponens and modus tollens. They can even agree on the general outlines of a method of conditional proof. But they have to disagree about the conditions to be applied. Philo can allow an unrestricted form of conditional proof, adopting a rule of the form \( \Gamma, A \vdash B \Rightarrow \Gamma \vdash A \rightarrow B \). Diodorus has to restrict \( \Gamma \) to propositions that are always true; Chrysippus, to propositions that are necessarily true. Without a precise characterization of the semantics for the conditional along Philonian, Diodoran, or Chrysippan lines, and without some form of conditional proof, establishing completeness for the full system would be hopeless.\(^5\)

The Stoic system is obviously incomplete even with respect to the other connectives. There is no way to exploit conjunctions; we cannot get from \( A \land B \) to \( A \). There is also no way to introduce disjunctions; we cannot go from \( A, \lnot B \) to \( \oplus(A, B) \). As a result, we have no way of pulling apart and then putting together conjunctions and disjunctions, so we cannot prove that either is commutative. We may define conjunction in terms of disjunction as follows: \( \land(A_1, \ldots, A_n) \leftrightarrow \oplus(A_1, \ldots, A_n, B, \lnot B) \), where \( B, \lnot B \) are not among \( A_1, \ldots, A_n \). The equivalence, however, is not provable. Given \( \oplus(\lnot A_1, \ldots, \lnot A_n, B, \lnot B) \), it is possible to prove \( \land(A_1, \ldots, A_n) \), but the other direction is impossible.

If we remedy those obvious defects, we do obtain a complete system. Restrict the language to the connectives \( \lnot, \land, \lor, \oplus \). Adopt as axioms (where \( i \leq n \)):

1. \( \lnot \land (A_1, \ldots, A_n), A_i \vdash \lnot \land (A_1, \ldots, A_{i-1}, \ldots, A_{i+1}, \ldots, A_n) \)
2. \( \land(A_1, \ldots, A_n) \vdash A_i \)
3. \( \oplus(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n), \lnot A_i \vdash \oplus(A_1, \ldots, A_n) \)
4. \( \oplus(A_1, \ldots, A_n), A_i \vdash \land(\lnot A_1, \ldots, \lnot A_{i-1}, \lnot A_{i+1}, \ldots, \lnot A_n) \)
5. \( \oplus(A_1, \ldots, A_n), \lnot A_i \vdash \oplus(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n) \)

\( A \rightarrow (A \rightarrow B), A \vdash B \) (Sextus, Adv. Math. VIII, 230–233)
\( (A \land B) \rightarrow C, \lnot C, A \vdash \lnot B \) (234–236)
\( A \lor B \rightarrow C, \lnot A, \lnot B \rightarrow C \) (Sextus, Hyp. Pyrrh. 169)
\( A \rightarrow B, A \vdash \lnot B \rightarrow A \)
\( A \rightarrow \lnot A, \lnot A \vdash \lnot A \rightarrow A \)
\( A \lor \lnot A \rightarrow A \) (Kneale and Kneale 168)
\( A \lor \lnot A, \lnot A \vdash \lnot A \) (Diogenes Laertius 7.194, Philodemus Sign. PHerc. 1065, XI.26–XII.14)

\(^5\)This has not stopped people from trying; see Becker 1957, Kneale and Kneale 1962, and Mueller 1979, all of whom interpret Stoic connectives as binary and adopt a Philonian reading of the conditional. In each case, however, some axioms and rules are added to make the system complete. Sometimes, these are artifacts of a Gentzen system that are extraneous to Stoic logic; sometimes, they are found in Stoic sources, but as theorems rather than basic axioms or rules.
Adopt indirect proof and cut as complex rules. Now, construct maximal consistent sets of formulas in standard Henkin fashion. The heart of the proof is to show that, for any maximal consistent set $\Gamma$, four proof-theoretic lemmas hold:

1. $\neg A \in \Gamma \iff A \notin \Gamma$. The construction process guarantees that either $A$ or $\neg A$ belongs to $\Gamma$; they appear at some stage of the enumeration. At the stage at which $A$ appears, either $A$ or $\neg A$ is placed in $\Gamma$. Say $\neg A \in \Gamma$ and $A \in \Gamma$. Then $\Gamma$ is inconsistent; contradiction.

2. $\Gamma \vdash A \Rightarrow A \in \Gamma$. Say $\Gamma \vdash A$ but $A \notin \Gamma$. Then, by the preceding lemma, $\neg A \in \Gamma$. But then $\Gamma \vdash A$ and $\Gamma \vdash \neg A$, so $\Gamma$ is inconsistent; contradiction.

3. $\land(A_1, \ldots, A_n) \in \Gamma \iff A_i \in \Gamma$ for each $i$ among $1, \ldots, n$. Say $A_i \in \Gamma$ for each $i$ among $1, \ldots, n$ but $\land(A_1, \ldots, A_n) \notin \Gamma$. Then $\neg \land (A_1, \ldots, A_n) \in \Gamma$. By Axiom 1, we may deduce $\neg \land (A_1, \ldots, A_n), \neg \land (A_2, \ldots, A_n)$, and so on, eventually reaching $\neg \land (A_n) = \neg A_n$. Since $\Gamma$ is closed under the axioms and rules, that implies that $A_n, \neg A_n \in \Gamma$, contradicting the previous lemma.

   For the other direction: Say $\land(A_1, \ldots, A_n) \in \Gamma$. We can derive each $A_i$ by Axiom 2. So, $A_i \in \Gamma$ for each $i$ among $1, \ldots, n$.

4. $\oplus(A_1, \ldots, A_n) \in \Gamma \iff A_i \in \Gamma$ for exactly one $i$ among $1, \ldots, n$. Say $\oplus(A_1, \ldots, A_n) \in \Gamma$ but none of $A_1, \ldots, A_n \in \Gamma$. Then $\neg A_1, \ldots, \neg A_n \in \Gamma$, by the previous lemma. By Axiom 5, we can derive $\oplus(A_2, \ldots, A_n), \oplus(A_3, \ldots, A_n)$, and so on, until we reach $\oplus(A_2) = A_n$. But then $A_n, \neg A_n \in \Gamma$, contradicting the previous lemma. So, at least one of $A_1, \ldots, A_n \in \Gamma$. Suppose more than one belong to $\Gamma$. For convenience, but without loss of generality, assume that $A_i, A_j \in \Gamma$, $i < j$, where no other disjuncts belong to $\Gamma$. By two applications of Axiom 4, we obtain $\oplus(A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n)$, but, for all $k \leq n$ such that $k \neq i, j$, $\neg A_k \in \Gamma$. Once again, by repeated applications of Axiom 5, we derive a contradiction.

   For the other direction: Say $A_i \in \Gamma$ for exactly one $i$ among $1, \ldots, n$. Since, for $j \neq i, j \leq n, A_j \notin \Gamma$, $\neg A_j \in \Gamma$. Begin with $A_i = \oplus(A_i)$. By repeated applications of Axiom 3, we derive $\oplus(A_1, \ldots, A_n)$, so $\oplus(A_1, \ldots, A_n) \in \Gamma$.

Now, given a maximal consistent set $\Gamma$ and any proposition $A$, we construct an interpretation $\nu$ from atomic formulas into truth values such that $A \in \Gamma \iff \nu \models A$. We stipulate that, for all atomic formulas $A$, $\nu(A) = 1$ for all $A \in \Gamma$.

Assume the hypothesis for all propositions of complexity less than that of an arbitrary formula $A$. We must consider several cases:

1. $A = \neg B$. By hypothesis, $B \in \Gamma \iff \nu(B) = 1$. Say $\neg B \in \Gamma$. Since $\Gamma$ is consistent, $B \notin \Gamma$, so $\nu(B) = 0$. But then $\nu(\neg B) = 1$. For the other direction, assume that $\nu(\neg B) = 1$. Then, $\nu(B) = 0$, so $B \notin \Gamma$. Since $\Gamma$ is maximal, $\neg B \in \Gamma$.

2. $A = \land(B_1, \ldots, B_n)$. By hypothesis, $B_i \in \Gamma \iff \nu(B_i) = 1$ for each $i \in \{1, \ldots, n\}$. Say $\land(B_1, \ldots, B_n) \in \Gamma$. By the above lemma, $B_1, \ldots, B_n \in \Gamma$, so, by hypothesis, $\nu(B_1) = \ldots = \nu(B_n) = 1$. So, $\nu(\land(B_1, \ldots, B_n)) = 1$.  

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Say \( v(\land(B_1, ..., B_n)) = 1 \). By the truth definition for \( \land \), \( v(B_1) = ... = v(B_n) = 1 \). But then \( B_1, ..., B_n \in \Gamma \), so, by the above lemma, \( \land(B_1, ..., B_n) \in \Gamma \).

3. \( A = \oplus(B_1, ..., B_n) \). By hypothesis, \( B_i \in \Gamma \iff v(B_i) = 1 \) for each \( i \in \{1, ..., n\} \). Say \( \oplus(B_1, ..., B_n) \in \Gamma \). By the above lemma, exactly one of \( B_1, ..., B_n \) belongs to \( \Gamma \). But then \( v \) makes exactly one of \( B_1, ..., B_n \) true. So, by the truth definition, \( v(\oplus(B_1, ..., B_n)) = 1 \).

Say \( v(\oplus(B_1, ..., B_n)) = 1 \). By the truth definition for \( \oplus \), \( v \) makes exactly one of \( B_1, ..., B_n \) true. By hypothesis, then, exactly one of \( B_1, ..., B_n \) belongs to \( \Gamma \). But then by the above lemma \( \oplus(B_1, ..., B_n) \in \Gamma \).

We have constructed an interpretation making everything in \( \Gamma \) true, given \( \Gamma \)'s consistency. So, every consistent set of formulas is satisfiable. It follows that, if the set consisting of an argument's premises and the negation of its conclusion is unsatisfiable, it is inconsistent. So, if an argument is valid, its conclusion is provable from its premises.

It is surprising that the Stoics did not include rules of simplification and disjunction introduction. They may have taken the former for granted, but, given the extent of their concern for disjunction, it is unlikely they did the same for the latter. We have no record, however, of an argument that \( A, \neg B \vdash \oplus(A, B) \) is admissible in their system. Nor do we have any record of its inclusion among the axioms. The concept of a deduction system as structured around introduction and elimination or exploitation rules was evidently foreign to them.

3 Hypothetical Syllogisms

Aristotle refers briefly to hypothetical reasoning, but develops no theory of it in any surviving texts. His successor Theophrastus (371–287 BC), however, is reputed to have developed a theory of hypothetical syllogisms—in modern terms, a theory of the conditional. The nature of that theory, and whether it in fact existed, remains a subject of scholarly dispute.\(^6\) Alexander of Aphrodisias (fl. 200 AD) says that Theophrastus discussed the inference pattern commonly known today as hypothetical syllogism: \( A \rightarrow B, B \rightarrow C \vdash A \rightarrow C \). Such an interpretation, however, is not that of Alexander, who thinks of the variables as taking the place of terms rather than propositions. His example: "If man is, animal is; if animal is, substance is; if therefore man is, substance is" (\textit{In An. Pr.} 326, 20–327, 18). John Philoponus (490–570), however, gives this example of a simple hypothetical syllogism:

\begin{align*}
\text{If the sun is over the earth, then it is day.} \\
\text{If it is day, then it is light.} \\
\text{Therefore, if the sun is over the earth, then it is light. (Scholia to Ammonius, \textit{In An. Pr.}, Wallies, \textit{Praefatio}, xi, in Mates 1961, 129)}
\end{align*}

This treats the variables as taking the place of propositions. But Philoponus is not consistent about this; he goes on to substitute terms for the variables.

\(^6\)See, for example, Barnes 1984 and Speca 2001.
Alexander gives another form that combines hypothetical syllogism and contraposition: $A \rightarrow B, B \rightarrow C \vdash \neg C \rightarrow \neg A$. He also offers $A \rightarrow C, B \rightarrow \neg C + A \rightarrow \neg B$ and $A \rightarrow B, \neg A \rightarrow C + \neg B \rightarrow C$ (and $\neg C \rightarrow B$).

Alexander reports that Theophrastus introduced other forms of argument that might be considered propositional: by subsumption of a third term, from a disjunctive premise, from denial of a conjunction, by analogy or similarity of relations, and by degrees of a quality (In Ar. An. Pr. I Wallies 389ff.; see Kneale and Kneale 1962, 105). We do not know what Theophrastus produced concerning these forms, but Boethius (DSH 831) says that it was not very substantial.

Alexander, however, gives examples of modus ponens in propositional and generalized forms:

If the soul always moves, the soul is immortal.
The soul always moves.
Therefore the soul is immortal.

If what appears to be more sufficient for happiness is not in fact sufficient, neither is that which appears to be less sufficient.
Health appears to be more sufficient for happiness than wealth and yet it is not sufficient.
Therefore wealth is not sufficient for happiness. (In Ar. An. Pr. I, 265; Kneale and Kneale 1962, 106)

Here the substituends for the variables are plainly propositions.

The ambiguity about the role of variables in this early theory of hypothetical syllogisms is not merely evidence of confusion. Theophrastus arranges his argument forms into three figures, thinking of them as closely analogous to syllogisms with universal premises and a universal conclusion. He thinks of $A \rightarrow B, B \rightarrow C + A \rightarrow C$, for example, as analogous to Barbara: Every $S$ is $M$, Every $M$ is $P$, therefore Every $S$ is $P$. If so, however, his theory of hypothetical syllogisms outstrips Aristotle’s theory of categorical syllogisms, for $A \rightarrow B, B \rightarrow C + \neg C \rightarrow \neg A$ would be analogous to Every $S$ is $M$, Every $M$ is $P$, therefore Every non$P$ is non$S$, which goes beyond Aristotle’s patterns. Not until Boethius would contraposition be recognized as a legitimate immediate inference and infinite terms such as non$P$ be given their due. The analogy between conditionals and universals is suggestive, though no one before Boole, Peirce, and Frege would fully exploit it.

Suppose, however, we were to think of conditionals $A \rightarrow B$, along Theophrastus’s lines, as analogous to universal affirmatives, in effect having truth conditions of the form “Every case in which $A$ is a case in which $B$.” Call this the universality thesis. We might also think of particular propositions as corresponding to conjunctions, which are understood to be commutative. And we might think of existential presuppositions as corresponding to distinct assertions. (One might think that ‘Every $S$ is $P$’ entails ‘Some $S$ is $P$,’ for example, but it is implausible to think that $S \rightarrow P$ entails $S \land P$ without the additional premise $S$.) Valid syllogistic forms would then generate a set of inference rules for conditionals. Permitting substitutions of negated propositions allows for some simplification:
Barbara, Celarent: $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

Cesare: $A \rightarrow B, C \rightarrow \neg B \vdash A \rightarrow \neg C$

 Camestres: $A \rightarrow \neg B, C \rightarrow B \vdash A \rightarrow \neg C$

 Calemes: $A \rightarrow \neg B, C \rightarrow A \vdash B \rightarrow \neg C$

 Darii, Ferio, Datisi, Disamis, Ferison, Bocardo, Dimatis: $A \land B, B \rightarrow C \vdash A \land C$

 Festino, Fresison: $A \land B, C \rightarrow \neg B \vdash A \land \neg C$

 Baroco: $A \land \neg B, C \rightarrow A \land \neg C$

 Barbari, Celaront, Bamalip: $A \rightarrow B, B \rightarrow C \vdash A \land C$

 Cesaro: $A, A \rightarrow B, C \rightarrow \neg B \vdash A \land \neg C$

 Camestros: $A, A \rightarrow \neg B, C \rightarrow B \vdash A \land \neg C$

 Calemos: $A, B \rightarrow \neg A, C \rightarrow B \vdash A \land \neg C$

 Darapti, Felapton: $A, A \rightarrow B, A \rightarrow C \vdash B \land C$

 Fesapo: $A, A \rightarrow B, C \rightarrow \neg A \vdash A \land \neg C$

 Surprisingly, only one of the three forms mentioned by Alexander is among these. If we were to assume double negation and contraposition, this would of course become far simpler, and all three of Alexander’s forms would be readily derivable:

 Hypothetical Syllogism: $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

 Conjunctive Modus Ponens: $A \land B, B \rightarrow C \vdash A \land C$

 Chaining: $A, A \rightarrow B, B \rightarrow C \vdash A \land C$

 Conjunctive Consequents: $A, A \rightarrow B, A \rightarrow C \vdash B \land C$

 In any case, Theophrastus’s theory of the conditional is plainly incomplete if we construe it as an account of strict implication, even if we credit him with an understanding of modus ponens and the other principles above. There is no way to derive a conditional from non-conditional premises. So, in particular, there is no way to derive $A \rightarrow A$. Nor is there any way to get from $\neg A \rightarrow A$ to $A$, or from $A \rightarrow \neg A$ to $\neg A$. Of course, Theophrastus may well have rejected all these inferences on Aristotelian grounds, insisting that antecedents must be possible for a conditional to count as true. Even in that case, however, the theory is incomplete, for $\neg (A \rightarrow \neg A)$ is not a theorem.

 Boethius (480–524?), a Roman nobleman born just after the fall of Rome, translated Aristotle’s logical works into Latin, wrote commentaries on them, and wrote several important logical works of his own, including one on hypothetical syllogisms. His translations of the Categories and De Interpretatione, as well as his other works, served as the chief sources of information about ancient logic for medieval thinkers until the thirteenth century.
Boethius’s De Syllogismo Hypothetico uses schemata such as *Si est A, est B*, which suggest a term interpretation but are ambiguous between that and a propositional interpretation (reading ‘*est*,’ in the latter case, as “it is the case that”). He speaks of hypotheticals as complex propositions containing other propositions as components; what is unclear is simply whether *A* or ‘*est A*’ (or both) should be taken as a proposition.

Boethius discusses the truth conditions of conditionals, distinguishing accidental from natural conditionals. An accidental conditional is one whose consequent is true when its antecedent is true. Thus, he says, ‘When fire is hot, the sky is round’ is true, because, “at what time fire is hot, the sky is round” (DSH 835). A natural conditional expresses a natural consequence, as in “when man is, animal is.” His temporal language suggests a Diodoran reading of the conditional, but it is not clear whether ‘at what time’ is meant as a quantifier.

The term Boethius uses for a conditional connection, accidental or natural, is *consequentia*, which he uses to translate Aristotle’s terms *akolouthesis* and *akolouthia*, “following from” (Kneale and Kneale 1962, 192).

Boethius also has some interesting things to say about negations. “Every negation is indeterminate (infinita),” he says (DSH 1.3.2–3); negation can separate “contraries, things mediate to contraries, and disparates”—that is, things that are not incompatible but merely different from one another.

His truth conditions for disjunction treat it as inclusive, but perhaps modal:

> The disjunctive proposition that says ‘either *A* is not or *B* is not’ is true of those things that can in no way co-exist, since it is also not necessary that either one of them should exist; it is equivalent to that compound proposition in which it is said: ‘if *A* is, *B* is not.’ (DSH 875)

The equivalence (¬*A* ∨ ¬*B* ⇔ *A* → ¬*B*) suggests either a Philonian reading for the conditional or a modal reading for disjunction.

Boethius lists valid inference patterns for conditionals:

1. *A* → *B*, *A* ⊢ *B*
2. *A* → *B*, ¬*B* ⊢ ¬*A*
3. *A* → *B*, *B* → *C* ⊢ *A* → *C*
4. *A* → *B*, *B* → *C*, ¬*C* ⊢ ¬*A*
5. *A* → *B*, ¬*A* → *C* ⊢ ¬*B* → *C*
6. *A* → ¬*B*, ¬*A* → ¬*C* ⊢ *B* → ¬*C*
7. *B* → *A*, ¬*C* → ¬*A* ⊢ *B* → ¬*C*
8. *B* → *A*, ¬*C* → ¬*A* ⊢ *B* → *C*
9. *A* ⊕ *B* ⊢ ¬*B*, ¬*A* → *B*, ¬*B* → *A*, *B* → ¬*A*
10. ¬*A* ∨ ¬*B* ⇔ *A* → ¬*B*
Boethius uses ‘or’ (aut) for both 9 and 10, but it is important to distinguish them if conditionals are not to become biconditionals. This theory extends that of Theophrastus, by taking account of contraposition and infinite terms.

4 Early Medieval Theories

Boethius clearly had access to Aristotle’s logical works. So did John Philoponus (490–570), one of our chief sources for Stoic logic, and Simplicius (490–560), both of whom wrote commentaries on Aristotle’s logical works. Shortly thereafter, however, the works of Aristotle, except for the Categories and De Interpretatione, disappeared.

After the sixth century, logicians knew logic chiefly through the works of Porphyry and Boethius. The result was the logica vetus—the Old Logic.

The Old Logic’s chief focus was Aristotelian syllogistic, as presented by Boethius. Propositional logic remained in the background. The Old Logicians nevertheless devote some attention to the logical connectives and have some interesting things to say about them.

The central concept of propositional logic is that of a proposition. The Old Logic has a standard definition, taken from Boethius: “propositio est oratio verum falsumve significans” (A proposition is a statement signifying truth or falsehood). Sometimes it appears in the form “propositio est oratio verum vel falsum significans indicando” or “propositio est oratio que cum indicatione significat verum falsumve,” which might be translated, “a proposition is a statement that purports to signify truth or falsehood.”

The former definition guarantees bivalence; the latter allows for the possibility that some propositions purport to signify a truth value but do not by virtue of being in a mood other than the indicative. “Sortes to run” and “Plato to dispute” are examples. They are propositions, figuring as components in compound propositions such as “Socrates wants Sortes to run” and “Thrasymachus allows Plato to dispute,” but they do not in such a context signify a truth value in the way they would in the indicative mood.

Old Logic texts distinguish various kinds of compound propositions (Ars Emmerana, 159; cf. Ars Burana, 190–191):

- **Conditionals**: If you are running, you are moving.

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7Missing from this list is Boethius’s most famous thesis, that ¬((A → B) ∧ (A → ¬B))). Such a thesis may or may not be defensible, but we fail to find it in Boethius. For discussion of this point, see Dühr 1951, Kneale and Kneale 1962, McCall 1966, Barnes 1981, and Martin 1991. Boethius’s thesis might be thought to follow from Aristotle’s idea that true conditionals have possible antecedents; the truth of A → B and A → ¬B would then require the possible truth of A and thus the possible truth of B ∧ ¬B. But we see no evidence that Boethius follows Aristotle on this point. It would in fact be surprising if he holds that view, for the fifth and sixth theses above could then fail; nothing in the premises would guarantee the possibility of B or ¬B. In fact, contraposition itself would be problematic, for A → B would not imply ¬B → ¬A, since A → B does not guarantee the possibility of ¬B. Perhaps this explains why Aristotle did not himself employ contraposition.

Boethius, De Topicis Differentiis, 2.22–23. This definition is ubiquitous, found in the Introductions Montane Minores, 18; Abbrevatio Montana, 79; De Arte Dialectica, 122; De Arte Dissersendi, 128; Ars Emmerana, 152; Ars Burana, 183; Introducciones Parisienses, 359; Logica “Cum Sit Nostra,” 419. It persists into the fourteenth century, appearing in Buridan (2001, 21).

9This formulation appears, for example, in the Dialectica Monacensis, de Rijk 1967, 468.
• **Locals:** You are sitting where I am standing.

• **Causals:** Because Tully is a man, Tully is capable of laughter.

• **Temporals:** While Socrates runs, Plato is moved.

• **Conjunctions:** Socrates is a man and Brunellus is a donkey.

• **Disjunctions:** Someone is running, or nothing is moved.

• **Adjuncts:** The master is reading so that the student might improve.

The Old Logic considers conjunction and disjunction as grammatical conjunctions formed with the words ‘and’ and ‘or,’ respectively. They state concise truth conditions for both. A conjunction is true if and only if all (or both) its conjuncts are true; a disjunction is true if and only if some disjunct or other is true.\(^{10}\) This is disjunction in the modern, inclusive sense. Only in the *Dialectica Monacensis* is it distinguished from exclusive disjunction.\(^{11}\)

The Old Logicians spend more time on hypothetical propositions, that is, conditionals, which they define as statements formed with the conjunction ‘if.’\(^{12}\) Interestingly, the monks of St. Genevieve use subjunctive conditionals as their paradigm examples: “If to be blind and blindness were the same thing, they would be predicated of the same thing” and “If you had been here, my brother would not have died.”\(^{13}\) They take these to show that not all propositions have subject-predicate form.

Old Logicians generally do not attempt to state truth conditions for hypotheticals. The *Ars Burana*, however, does: “Every conditional is true whose antecedent cannot be true without the consequent, as ‘if Socrates is a man, Socrates is an animal.’ Also, every conditional is false whose antecedent either can or could or will be able to be true without the consequent, as ‘if Socrates is a man, then Socrates is a donkey’” (191). The mixture of modals and tense operators here is curious, suggesting a combination of Diodoran and Chrysippan ideas. But it appears to be equivalent to a strict conditional reading. An alternative account of truth conditions appears in the *Logica “Cum Sit Nostra,*’ which treats conditionals as true when the consequent is understood in the antecedent (*quando consequens intelligitur in antecedente*, 425). This appears to be stronger, requiring that the necessity in question be analytic. Another alternative appears in the *Dialectica Monacensis*, which begins with a strict conditional account—“to posit [the truth of] the antecedent it is necessary to posit [the truth of] the consequent” (484–485)—but then adds *vel saltim probabile*, “or at least probable.” The example is ‘If this is a mother, she loves.” The overall account, then, is that the truth of

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\(^{10}\) See, for example, *Ars Burana*, 191, *Logica “Cum Sit Nostra,”* 425–426.

\(^{11}\) That text treats disjunctions as ambiguous between a reading on which one disjunct is true and the other false and another (called subdisjunction) on which one or both is true (485).

\(^{12}\) Old Logic texts seem to equivocate on hypothetical propositions, sometimes treating them as containing the word ‘if’ or an equivalent, thus amounting to conditionals in the contemporary sense, and sometimes treating them as including conjunctions, disjunctions, and a variety of other propositions, in which case ‘hypothetical’ seems equivalent to ‘compound’ or ‘molecular.’ See, e.g., *Ars Emmerana* 158–159, which does both in a span of three paragraphs.

\(^{13}\) *Introductiones Montane Minores*, 39; the former is from Aristotle’s *Categories* (10), the latter from John 11:21.
the antecedent makes the truth of the consequent probable. That intriguing idea, which appears to stem from William of Champeaux (1070–1122), then disappears for several hundred years.

The *Introductiones Norimbergenses* defines a hypothetical as saying something about something *under a condition* (*sub conditione*; 140–141). This is ambiguous between an ordinary view on which a conditional makes an assertion that something holds under a condition and a conditional assertion view that a conditional makes an assertion only conditionally. The latter view would make it difficult to maintain, however, that hypotheticals are propositions, for propositions are, in that work, always true or false.

Old Logicians divide hypotheticals into various kinds. *Simple* hypotheticals have no hypotheticals as components; *composite* hypotheticals do. The Old Logicians take it for granted that there can be embedded conditionals. Indeed, any hypothetical associated with an argument containing as hypothetical premise or conclusion is bound to be composite. Though Old Logicians do not for the most part speak explicitly about associated conditionals, they are aware of them. “An argument can be transformed into a conditional proposition, just as a conditional proposition can be taken as an argument” (*Ars Emmerana*, 164.) The monks of St. Genevieve give an example: “If every man is an animal and every animal is a substance, then every man is a substance” (*Introductiones Montane Minores*, 40). This, they hold, is neither simple nor composite, for it has no hypothetical as a component, but it has a conjoined antecedent. Evidently the definition of simplicity they intend is this: a conditional is simple if and only if its antecedent and consequent are both atomic propositions.

Hypotheticals, strictly speaking, contain the connective ‘if,’ but there are other connected (*connexa*) propositions that express very similar relationships, such as those formed with the connectives ‘when,’ ‘as often as,’ ‘as long as,’ ‘while,’ and so on.

Old Logicians are familiar with modus ponens from reading Boethius. The monks follow Boethius in expressing it in a hybrid form:

\[
\begin{align*}
\text{If there is } A, \text{ there is } B \\
\text{There is } A \\
\text{Therefore, there is } B
\end{align*}
\]

They give as an example “As often as Socrates reads, he speaks; but Socrates is reading; therefore, Socrates is speaking.” The inference rule they have in mind, then, is a generalized form of modus ponens, one which, in modern form, we might express as \( \forall x(Ax \rightarrow Bx), Ac \vdash Bc \). In other places, however, they state a straightforward version: “in every hypothetical proposition, if the antecedent is true, the consequent may be inferred” (45).

The monks discuss negations of conditionals and, in particular, the hypothesis, which they attribute to Boethius, that the negation of a conditional is equivalent to a conditional with a negated consequent: \( \lnot(A \rightarrow B) \iff (A \rightarrow \lnot B) \). They carefully distinguish the affirmative conditionals \( A \rightarrow B \) and \( A \rightarrow \lnot B \) from the negated conditionals \( \lnot(A \rightarrow B) \) and \( \lnot(A \rightarrow \lnot B) \), and observe that only the former license modus ponens

\[14\text{Introductiones Montane Minores, 43. As in Boethius, this is ambiguous between, e.g., “There is } A \text{” and “It is the case that } A \text{.”} \]
ponens (46). They draw an analogy with necessity to make their point about scope: “If a man is living, it is necessary for him to have a heart.” Can we apply modus ponens? Suppose a man is living. Can we infer that he necessarily has a heart? If the necessity attaches to the consequent alone, we could, but we would be drawing a false conclusion, indicating that the original conditional is false. If the necessity attaches to the entire hypothetical, however, we can infer only that he has a heart. Similarly, if the negation has scope only over the consequent, we can apply modus ponens. But we cannot if the negation has scope over the entire conditional. So, \( \neg(A \rightarrow B) \) and \( A \rightarrow \neg B \) are not equivalent; only the latter licenses the move from \( A \) to \( \neg B \).

5 Later Medieval Theories

The New Logic arose in the thirteenth century with the rediscovery of Aristotle’s logical works and the textbooks of William of Sherwood, Lambert of Auxerre, and Peter of Spain. Peter gives clear truth conditions for conditionals, conjunctions, and disjunctions, with separate clauses for truth and falsehood. He treats conjunction and disjunction as multigrade, and his understanding of disjunction is inclusive:

For the truth of a conditional it is required that the antecedent cannot be true without the consequent, as in, ‘If it’s a man, it’s an animal.’ From this it follows that every true conditional is necessary, and every false conditional is impossible. For falsehood it suffices that the antecedent can be true without the consequent, as ‘if Sortes is, he is white.’

For the truth of a conjunction it is required that all parts are true.... For falsehood it suffices that some part or another be false....

For the truth of a disjunction it suffices that some part or other is true.... It is permitted that all parts be true, but it is not so properly [felicitously].... For falsehood it ought to be that all parts are false.... (Peter of Spain, Tractatus I, 17, 9–10)

This is the strict conditional analysis of conditionals, which had been championed by Abelard (1079–1142) as well as most twelfth-century texts. Unlike most of the Old Logicians, however, Peter follows Abelard (Dialectica, 160, 279) to draw the conclusion that all conditionals are necessarily true or necessarily false. That is intriguing evidence that his background conception of modality is S5. He analyzes \( A \rightarrow B \), essentially, as \( \Box(A \supset B) \), and infers from that \( \Box(A \rightarrow B) \), i.e., \( \Box\Box(A \supset B) \). Similarly, from \( \Diamond(A \land \neg B) \) he infers \( \Box\Diamond(A \land \neg B) \).

Peter’s conception of the connectives persists into the fourteenth century, though his multigrade conception begins to yield to a binary conception. William of Ockham (1287–1347), for example, gives similar truth conditions for conjunction, but takes it as binary in the positive half of the definition and multigrade in the negative half:

Now for the truth of a conjunctive proposition it is required that both parts be true. Therefore, if any part of a conjunctive proposition is false, then the conjunctive proposition itself is false (1980, 187).
John Buridan (1290s–1360?) states the same positive condition, observing that an analogous condition holds for conjunctions of more than two terms (2001, 62–63).

Ockham goes further than Peter of Spain in three ways, however. First, he thinks about the interaction of conjunction and modality. For a conjunction to be necessary, both parts must be necessary. Second, he considers the negation of a conjunction, which is equivalent to a disjunction: \( \neg(A \land B) \equiv (\neg A \lor \neg B) \). Third, he formulates inference rules for conjunction. “Now it is necessary to note that there is always a valid consequence from a conjunctive proposition to either of its parts” (1980, 187). He warns that one cannot go from just one conjunct to the conjunction, except in the case in which that conjunct entails the other, but fails to point out that one can move from both conjuncts to the conjunction.

Ockham and Buridan follow Peter in treating disjunction as inclusive; for its truth he requires merely that some disjunct be true. The negation of a disjunction, Ockham observes, is equivalent to a conjunction: \( \neg(A \lor B) \equiv (\neg A \land \neg B) \). He formulates inference rules for disjunction as well, stating rules for addition—licensing the move from one disjunct to the disjunction—and disjunctive syllogism: “Socrates is a man or a donkey; Socrates is not a donkey; therefore Socrates is a man” (1980, 189).

Ockham follows Peter, too, in his account of the truth conditions of conditionals. “A conditional proposition is true,” he says, “when the antecedent entails the consequent and not otherwise” (1980, 186). He defers a fuller discussion to his tract on consequences.

Buridan seems to accept a similar condition: “the antecedent cannot be true along with the consequent’s not being true, provided both exist simultaneously, or even better, that it is not possible for things to be in the way signified by the antecedent without their being in the way signified by the consequent” (2001, 62). He draws attention, however, to two exceptions. First, this does not apply to consequences as-of-now (consequentia ut nunc), for they depend on additional contingent information. His example: “Gerard is with Buridan; therefore he is in rue du Fouarre” (2001, 62). This is not a true logical consequence but an enthymeme requiring an additional contingent premise. Second, they do not apply to future contingents, and in particular promises. ‘If you visit me, I’ll give you a horse’ does not require that the visit entail the horse-giving, for of course the promise might be broken. If you visit, and I give you the horse, I have kept my promise; thus, ‘If you visit me, I’ll give you a horse’ turned out to be true. Buridan is noticing that sometimes \( A \) and \( B \) suffice for the truth of \( A \rightarrow B \). But that would not be true on Peter’s account, unless \( B \) was itself necessary.

The most striking development in fourteenth-century logic relevant to the connectives, however, is the theory of consequences. What is this a theory of? Boethius uses consequentia, as we have seen, for a conditional connection. But he is translating a term Aristotle uses for entailment. Abelard distinguishes these, using consequentia strictly for the former and consecutio for the latter. Abelard’s theory is thus a theory of conditionals. His thesis that conditionals are true if and only if the antecedent entails the consequent, however, makes it easy to run the two notions together. Even so great a logician as Buridan defines a consequence as a hypothetical composed of sentences

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15 Among those who follow him in this are Pseudo-Scot: “A consequence is a hypothetical proposition composed of an antecedent and a consequent by means of a conditional connective....” (In An. Pr. I, 10, 7).
joined by ‘if,’ ‘therefore,’ or their equivalent (1985, 1.3.2, 181). Fourteenth-century accounts, as we shall see, are best seen as theories of entailment.

Walter Burley (1275–1344), for example, outlines a theory of consequences that endorses both the rule that, in a good consequence, “the antecedent cannot be true without the consequent” (285, my emphasis) and that “the contradictory of a conditional proposition is equivalent to a proposition that signifies that the opposite of its consequent stands together with its antecedent” (297). For Burley, plainly, a consequence is not a conditional, but a relation of entailment. Burley’s theory, in other words, is a theory of consecution in Abelard’s sense. We shall accordingly represent statements of consequence as consecutions, that is, schemata of the form $A_1, \ldots, A_n \models B$.

The first four of Burley’s rules are these, which are those most pertinent to the connectives:

1. The antecedent cannot be true without the consequent: $A \models B \Rightarrow \neg \Diamond (A \land \neg B)$ (or, perhaps better, $A \models B \Rightarrow \Box (A \rightarrow B)$. He infers from this rule that what is impossible does not follow from what is contingent ($\Diamond A \land \neg \Diamond B \Rightarrow A \models B$) and that what is contingent does not follow from what is necessary ($\Box A \land \neg \Box B \Rightarrow A \models B$).

2. Whatever follows from the consequent follows from the antecedent: $A \models B, B \models C \Rightarrow A \models C$. The relation of entailment, then, is transitive. Burley considers several entertaining potential objections—e.g., “the uglier you are, the more you embellish yourself; the more you embellish yourself, the more attractive you are; therefore, the uglier you are, the more attractive you are”—which, he argues, are based on equivocations. He also draws several corollaries: What follows from the consequent and antecedent follows from the antecedent by itself ($A \models B, A, B \models C \Rightarrow A \models C$) and what follows from the consequent with some addition follows from the antecedent with the same addition ($A \models B, (B \land C) \models D \Rightarrow (A \land C) \models D$).

3. The contradictory of the antecedent follows from the contradictory of the consequent ($A \models B \Rightarrow \neg B \models \neg A$). Burley points out that Aristotle uses this principle in reducing certain syllogisms to first figure by reductio.

4. The formal element affirmed in one contradictory must be denied in the other. Burley uses this as a generalization covering instances such as $\neg \Diamond (A \land B) \Leftrightarrow (\neg A \lor \neg B), \neg (A \lor B) \Leftrightarrow (\neg A \land \neg B)$, and $\neg (A \rightarrow B) \Leftrightarrow (A \land \neg B)$.

Buridan and Albert of Saxony (1316–1390) offer a more comprehensive set of rules for consequences, all of which they derives from a definition of consequence like Burley’s:

1. From the impossible everything follows ($\neg \Diamond A \Rightarrow A \models B$).

2. The necessary follows from anything ($\Box A \Rightarrow B \models A$).

3. Every proposition implies anything whose contradictory is incompatible with it ($\neg \Diamond (A \land \neg B) \Rightarrow A \models B$).
4. The contradictory of the consequent implies the contradictory of the antecedent
\((A \models B \Rightarrow \neg B \models \neg A)\).

5. Transitivity \((A \models B \land B \models C \Rightarrow A \models C)\).

6. It is impossible for the false to follow from the true, the possible from the impossible, or the contingent from the necessary \((A \land \neg B \Rightarrow A \not\models B; \neg \Box A \land \Box B \Rightarrow A \not\models B)\). Albert deduces that what implies the false is false; the impossible, impossible; and the unnecessary, unnecessary \((A \models B \Rightarrow (\neg B \Rightarrow \neg A)); A \models B \Rightarrow (\neg \Box B \Rightarrow \neg \Box A))\).

7. If one thing follows from another together with necessary propositions, it follows from that other alone \((\Box C, A \land C \models B \Rightarrow A \models B)\).

8. Contradictions imply anything \((A, \neg A \models B)\). It follows that anything that implies a contradiction implies anything \((A \models B \land \neg B \Rightarrow A \models C)\).

The fourteenth-century theory of consequences foreshadows abstract logic, which characterizes the relation of entailment from an abstract point of view. Surprisingly, Ockham, Burley, Buridan, and Albert do not remark on the reflexivity of entailment. They do notice transitivity, however, and the behavior of necessary and impossible propositions lets them play the role of top and bottom elements in a lattice.

The last great medieval logician was Paul of Venice (1369–1429), who writes extensively about the propositional connectives. Paul recognizes that conjunction and disjunction apply to terms as well as propositions and discusses each use in detail.

Paul presents a definition of proposition that deviates from the usual medieval definition, characterizing it as a well-formed and complete mental sentence \((\text{congrua et perfecta enuntiatio mentalis})\) that signifies truth or falsehood. He distinguishes between a proposition (say, \(A\)) and its adequate significate \((\sigma A)\). Paul gives four rules for truth and falsehood. Where a diamond stands for consistency:

1. \(T[\sigma(A)] \land \Diamond T[A] \Rightarrow T[A]\)
2. \(T[A] \Rightarrow T[\sigma(A)]\)
3. \(F[\sigma(A)] \Rightarrow F[A]\)
4. \(F[A] \land \Diamond F[\sigma(A)] \Rightarrow F[\sigma(A)]\)

The complexity introduced by the first and fourth rules is designed to handle insolvability such as the Liar paradox, which end up false, but in a peculiar way—because the consistency clause fails. Suppose \(\Diamond T[A]\) and \(\Diamond F[\sigma(A)]\). Then these rules imply that \(T[A] \leftrightarrow T[\sigma(A)]\) and \(F[A] \leftrightarrow F[\sigma(A)]\). Assume that the adequate significates observe principles of bivalence and noncontradiction, so that \(T[\sigma(A)] \leftrightarrow \neg F[\sigma(A)]\). Consider ‘This sentence is false,’ which Paul could construe in one of two ways:

- \(T[L] \leftrightarrow F[L]\). Suppose \(T[L]\) and \(F[L]\); then \(T[\sigma(L)]\) and \(\Diamond F[\sigma(L)] \Rightarrow F[\sigma(L)]\). But \(\neg F[\sigma(L)]\), so \(\neg \Diamond F[\sigma(L)]\). Suppose \(\neg T[L]\) and \(\neg F[L]\). Then \(\neg T[\sigma(L)]\) or \(\neg \Diamond T[A]\) and \(\neg F[\sigma(L)]\). But then \(T[\sigma(L)]\); so, \(\neg \Diamond T[A]\). So, we can deduce that \(\neg \Diamond F[\sigma(L)]\) or \(\neg \Diamond T[L]\).
• $T[L] \Leftrightarrow F[\sigma(L)]$. (In effect, this interprets the liar sentence as “The proposition this sentence signifies is false.”) Say $T[L]$ and $F[\sigma(L)]$. Then $T[\sigma(L)]$; contradiction. So, say $\neg T[L]$ and $\neg F[\sigma(L)]$. Then $T[\sigma(L)]$. Since $\neg T[L]$, $\neg \sigma T[L]$.

The upshot is that, on either interpretation, the liar sentence fails to designate consistently an adequate significate.

Paul’s account of conjunction as a relation between terms is rich and surprisingly contemporary in feel. He distinguishes collective and distributed (divisive) readings. A distributed reading entails a conjunctive proposition: “Socrates and Plato are running. Therefore Socrates is running and Plato is running” (1990, 54). A collective reading does not: “Socrates and Plato are sufficient to carry stone A. Therefore Socrates is sufficient to carry stone A and Plato is sufficient to carry stone A” (1990, 55). He advances three theses about contexts that trigger distributed readings:

1. A conjoint term or plural demonstrative pronoun that is subject of a verb having no term in apposition has distributed supposition. (These people are eating, drinking, and loving. Therefore this one is eating, drinking, and loving and that one is eating, drinking, and loving.)

2. A conjoint term or plural demonstrative pronoun that is subject of a substantival verb with a term in apposition has distributed supposition. (These are runners. Therefore this is a runner and that is a runner.)

3. A conjoint term or plural demonstrative pronoun that has supposition in relation to an adjectival verb that determines a verbal composite has distributed supposition. (Socrates and Plato know two propositions to be true. Therefore Socrates knows two propositions to be true and Plato knows two propositions to be true.)

Three other theses describe contexts that trigger collective readings:

1. A conjoint term or plural demonstrative pronoun that has supposition in relation to a substantival verb having a singular subject or object has collective supposition. (Thus, “some man is matter and form”; “Shield A is white and black” (1990, 62).)

2. A conjoint term or plural demonstrative pronoun that has supposition in relation to a substantival verb with a subject or object and a determinant of it has collective supposition. (“A and B are heavier than C.” It does not follow that A is heavier than C.)

3. A conjoint term or plural demonstrative pronoun that has supposition in relation to an adjectival verb that has a subject or object distinct from the conjoint term or demonstrative pronoun has collective supposition. (“These men know the seven liberal arts” (1990, 65).)

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16 We adopt the convention of writing “therefore” for inferences that do not follow.
17 1990, 55. Paul’s text has vel (‘or’) in the premise and et (‘and’) in the conclusion, but this is surely a mistake; the same connective should appear in premise and conclusion.
18 We should read ‘two’ as ‘at least two.’
Conjunction is multigrade, according to Paul; conjunctive propositions may consist of two, three, or more parts. His truth conditions are standard: “For the truth of an affirmative conjunctive proposition the truth of each part is necessary and sufficient” (1990, 90). Paul’s presentation goes on to discuss modality as well as conjunctive knowledge and belief. A conjunction is necessary if and only if all its conjuncts are necessary. To know a conjunction, one must know every conjunct. Similarly, to believe a conjunctive proposition is to believe each conjunct. Uncertainty about any conjunct suffices for uncertainty about the whole. Paul presents a rule of simplification (1990, 99) but, like his predecessors, fails to specify a rule for introducing conjunctions.

Disjunction, too, applies to terms as well as to propositions. His rule for collective and distributed readings of disjunctions is simple: disjunctions always receive distributed readings unless “a determinant giving confused supposition, or a demonstrative term, covers the connective” (1990, 121). Disjunctive propositions are true if and only if at least one disjunct is true. As with conjunction, he states conditions for the modal status of disjunctions as well as for disjunctive knowledge and belief.

Paul devotes the most attention, however, to conditionals. He reviews and rejects many accounts of the truth of conditionals, and despairs of giving a fully adequate and comprehensive account because of the wide variety of conditionals and conditional connections. His basic account, however, is that $A \rightarrow B$ is true if $A$ is incompatible with $\neg B$ (1990b, 12). He does articulate certain inference rules, including $A \rightarrow B \vdash \neg A \vee B$ (40). He shows that some conditionals are contradictory (“If you are not other than yourself, you are other than yourself”) and uses it to show that some conditionals make positive assertions, which he takes as an argument against a conditional assertion account (41–42).

What is most interesting, from our point of view, is that Paul maintains a sharp distinction between conditionals and entailment propositions (propositiones rationales), formed with connectives such as ‘hence’ or ‘therefore.’ He defines a valid inference as “one in which the contradictory of its conclusion would be incompatible with the premise” (80). Thus, $A \vdash B$ is valid if and only if $\neg \Diamond (A \land \neg B)$. This is plainly equivalent to his truth condition for conditionals, however, so it turns out that an inference from $A$ to $B$ is valid (which we will write as $A \models B$) if and only if $A \rightarrow B$ is true.

Paul offers a series of theses that amount to a theory of consequences, together with three principles about propositional attitudes $K$ (knowledge), $B$ (belief), $U$ (understanding), $N$ (denial), and $D$ (doubt):

1. $\neg B \vdash \neg A \Rightarrow A \vdash B$
2. $A \vdash B, A \Rightarrow B$
3. $A \vdash B, \Box A \Rightarrow \Box B$
4. $A \vdash B, \Diamond A \Rightarrow \Diamond B$
5. $A \vdash B, B \vdash C \Rightarrow A \vdash C$
6. $A \vdash B, \Diamond (A \land C) \Rightarrow \Diamond (B \land C)$
7. $A \vdash B, K(A \vdash B), U(A \vdash B), B(A) \Rightarrow B(B)$
8. \( A \models B, K(A \models B), U(A \models B), N(B) \Rightarrow N(A) \)

9. \( A \models B, K(A \models B), U(A \models B), D(B) \Rightarrow D(A) \lor N(A) \)

10. \( A \models B, K(A \models B), U(A \models B), K(A) \Rightarrow K(B) \)

11. \( A \models B, K(A \models B), U(A \models B), D(A) \Rightarrow \neg N(B) \)

The first five of these relate closely to rules advanced by Buridan and Albert of Saxony, but the remainder appear to be new. Knowledge and belief are closed under logical consequence, Paul holds, but only provided that the agent understands the inference and knows it to hold.

6 Leibniz’s Logic

Gottfried Wilhelm Leibniz (1646–1716), diplomat, philosopher, and inventor of the calculus, developed a number of highly original logical ideas, from his idea of a *characteristica universalis* that would allow the resolution of philosophical and scientific disputes through computation (“Calculemus; Leibniz 1849–1863, 7, 200) to his algebra of concepts (Leibniz 1982). The algebra of concepts concerns us here, though it falls within a logic of terms, because Leibniz later recognizes that it can be reinterpreted as an algebra of propositions.

Let \( A, B \), etc., be concepts, corresponding to the components of Aristotelian categorical propositions. Leibniz introduces two primitive operations on concepts that yield other concepts: negation (\( \bar{A} \)) and conjunction (\( AB \)). Negating terms to form so-called infinite terms was nothing new; the idea goes back at least to Boethius. But conjoining terms was Leibniz’s innovation. He also introduces two primitive relations among concepts: containment (\( A \supset B \)) and possibility/impossibility (\( \diamond A / \neg \diamond A \)). The former Leibniz intends to capture the relationship between the subject and predicate of a universal affirmative proposition. The latter is new. This collection allows us to define converse containment and identity:

- \( A \subset B \iff B \supset A \)
- \( A = B \iff A \supset B \land B \supset A \)

Leibniz proceeds to state a number of principles.\(^\text{19}\) Some of these principles—transitivity and contraposition, for example—are familiar from ancient and medieval logic. Many, however, are new. Leibniz appears to be the first logician to include principles of idempotence.

- *Idempotence:* \( A \supset A \)
- *Transitivity:* \( (A \supset B) \land (B \supset C) \Rightarrow A \supset C \)
- *Equivalence:* \( A \supset B \iff A = AB \)

\(^\text{19}\)Lenzen (2004, 14–15) contains a useful chart. We use \( \supset \) rather than \( \in \) for containment to preserve greater continuity with modern usage and to suggest an analogy with propositional logic.
• Consequent Conjunction: $A \supset BC \iff A \supset B \land A \supset C$

• Simplification: $AB \supset A$

• Simplification: $AB \supset B$

• Idempotence: $AA = A$

• Commutativity: $AB = BA$

• Double Negation: $\neg\neg A = A$

• Nontriviality: $A \neq \neg A$

• Contraposition: $A \supset B \iff \neg B \supset \neg A$

• Contrapositive Simplification: $\neg A \supset \neg AB$

• Contrary Conditionals: $[\neg\neg(A) \land A \supset B \Rightarrow A \supset \neg B$

• Strict Implication: $\neg\neg(AB) \iff A \supset B$

• Possibility: $A \supset B \land \neg A \Rightarrow \neg B$

• Noncontradiction: $\neg\neg(A\neg A)$

• Explosion: $A\neg A \supset B$

This system is equivalent to Boole’s algebra of classes (Lenzen 1984). But it actually makes little sense as a theory of categorical propositions, for it gives all universal propositions modal force. Later, however, in the Notationes Generales, Leibniz realizes that propositions can be viewed as concepts on the space of possible worlds. That allows him to “conceive all propositions as terms, and hypotheticals as categoricals....” (Leibniz 1966, 66, quoted in Lenzen 2004, 35). To facilitate a propositional interpretation of the above, we translate negations and conjunctions into modern symbols, and replace identity with the biconditional:

• Idempotence: $A \supset A$

• Transitivity: $(A \supset B) \land (B \supset C) \Rightarrow (A \supset C)$

• Equivalence: $A \supset B \iff A \equiv (A \land B)$

• Predicate Conjunction: $(A \supset B) \land C \iff (A \supset B) \land (A \supset C)$

• Simplification: $(A \land B) \supset A$

• Simplification: $(A \land B) \supset B$

• Idempotence: $(A \land A) \equiv A$

• Commutativity: $(A \land B) \equiv (B \land A)$

• Double Negation: $\neg\neg A \equiv A$
• **Nontriviality:** \( \neg (A \equiv \neg A) \)
• **Contraposition:** \( (A \supset B) \iff (\neg B \supset \neg A) \)
• **Contrapositive Simplification:** \( \neg A \supset \neg (A \land B) \)
• **Contrary Conditionals:** \( ([\diamondsuit (A) \land (A \supset B)] \Rightarrow \neg (A \supset \neg B)) \)
• **Strict Implication:** \( \neg ^\land (A \land \neg B) \iff (A \supset B) \)
• **Possibility:** \( ((A \supset B) \land \diamondsuit A) \Rightarrow \diamondsuit B \)
• **Noncontradiction:** \( \neg \diamondsuit (A \land \neg A) \)
• **Explosion:** \( (A \land \neg A) \supset B \)

It remains to interpret the thick arrows and double arrows. Are these to be understood as object language conditionals and biconditionals, as Lenzen 2004 believes? If so, Leibniz offers a set of axioms, but without any rules of inference. Perhaps we should interpret them as metatheoretic, and construe the schemata in which they appear as rules of inference. That does not leave us in much better position, however, for we still lack modus ponens.

There is another possibility. Leibniz interprets his propositional logic as a logic of entailment: \( A \supset B \) is true if and only if \( B \) follows from \( A \) (Leibniz 1903, 260, 16: “ex A sequi B”). So, we might replace both the conditional and implication with an arrow that can be read either way. Leibniz does not distinguish object from metalanguage. Indeed, the Stoic-medieval thesis that an argument is valid if and only if its associated conditional is true makes it difficult to see the difference between the conditional and implication. So, we might think of all conditionals or biconditionals as permitting interpretation as rules of inference as well as axioms. We might, in other words, think of this as something like a theory of consequences in the fourteenth-century sense. That effectively builds modus ponens into the system by default, since \( A \rightarrow B \) can be read as a conditional or as a rule licensing the move from \( A \) to \( B \). Nevertheless, there seems to be no way to get from \( A \) and \( B \) to \( A \land B \); a rule of conjunction would have to be added separately.

• **Idempotence:** \( A \rightarrow A \)
• **Transitivity:** \( ((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C) \)
• **Equivalence:** \( (A \rightarrow B) \iff (A \leftrightarrow (A \land B)) \)
• **Consequent Conjunction:** \( (A \rightarrow (B \land C)) \iff ((A \rightarrow B) \land (A \rightarrow C)) \)
• **Simplification:** \( (A \land B) \rightarrow A \)
• **Simplification:** \( (A \land B) \rightarrow B \)
• **Idempotence:** \( (A \land A) \rightarrow A \)
• **Commutativity:** \( (A \land B) \leftrightarrow (B \land A) \)

29
• **Double Negation**: \( \neg\neg A \leftrightarrow A \)

• **Nontriviality**: \( \neg (A \leftrightarrow \neg A) \)

• **Contraposition**: \( (A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A) \)

• **Contrapositive Simplification**: \( \neg A \rightarrow \neg (A \land B) \)

• **Contrary Conditionals**: \( (\neg \neg A \land A \rightarrow B) \rightarrow \neg (A \rightarrow \neg B) \)

• **Strict Implication**: \( \neg \neg (A \land \neg B) \leftrightarrow (A \rightarrow B) \)

• **Possibility**: \( (A \rightarrow B) \land \Diamond A \rightarrow \Diamond B \)

• **Noncontradiction**: \( \neg \Diamond (A \land \neg A) \)

• **Explosion**: \( (A \land \neg A) \rightarrow B \)

Lenzen (1987) shows that this system is approximately Lewis’s S2°, although, as he points out, Leibniz surely accepted stronger principles about modality than this calculus represents.

It is striking, given the medieval consensus about inclusive disjunction, that Leibniz’s calculus contains no such connective. Nor does it contain some principles familiar from theories of consequences—for example, the principle that the necessary does not imply the contingent (in this notation, that \( \neg \Diamond (\neg \neg A \land A \rightarrow B) \rightarrow \neg \Diamond \neg B \)), or that the necessary follows from anything \( \neg \Diamond \neg A \rightarrow (B \rightarrow A) \), though the latter follows from strict implication, contraposition, contrapositive simplification, and possibility. Leibniz appears to have been constructing his algebra of concepts and the resulting propositional logic on a foundation independent of late medieval developments.

### 7 Standard Modern-Era Logic

Kant famously states in the preface to the second edition of the *Critique of Pure Reason* that “it is remarkable that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine”. This declaration captures much of the mood toward logic during the modern era. From the Renaissance through the beginning of the nineteenth century substantial advances in logic were rare, and typically had little influence when they did occur. Many significant philosophers of this period were dismissive of logic, and discussions of logic tended to be largely displaced by discussions of the psychology of belief formation and revision.

Logic textbooks of this period for the most part follow a common recipe. First, a theory of terms is given. The core of this theory is typically a canonical articulation of sentences into a subject-copula-predicate format:

Each proposition [contains] two terms; of these terms, that which is spoken of is called the subject; that which is said of it, the predicate; and these two are called the terms, or extremes, because, logically, the subject is placed first, and the predicate last; and, in the middle, the copula, which indicates the act of judgment, as by it the predicate is affirmed or denied.
of the subject. The copula must be either IS or IS NOT; which expres-
sions indicate simply that you affirm or deny the predicate, of the subject.
(Whately 1825, 85)

In every proposition there are three parts, viz. a Subject, denoting the
thing spoken of; a Predicate, denoting that which is asserted of it; and a
Copula, or the verb, which connects them by affirmation or denial. . . . The
Subject and predicate, though they are not always single words, are to be
considered as simple terms; for, as in Grammar a whole sentence may be
the nominative case to a verb, so in Logick a whole sentence may be the
subject or predicate of a proposition. . . . The copula is always the verb
substantive am, is, are, am not, &c. (Benthem 1773, 36)

This articulation is often accompanied with some discussion of how to massage recal-
citrant cases into the proper format:

It is worth observing, that an infinitive (though it often comes last in
the sentence) is never the predicate, except when another infinitive is the
subject: e.g. “I hope to succeed;” i.e. “to succeed (subj.) is what I hope
(pred.)” (Whately 1825, 87)

The theory of terms is followed by a theory of propositions. The theory of propo-
sitions tends ultimately to center on the Aristotelian division of propositions into
the universal affirmative (A), universal negative (E), existential affirmative (I), and existen-
tial negative (O). However, along the way to this taxonomy, various forms of molecular
propositions are often identified:

Propositions considered merely as sentences, are distinguished into
“categorical” and “hypothetical”. The categorical asserts simply that the
predicate does, or does not, apply to the subject. . . . The hypothetical
makes its assertion under a condition, or with an alternative; as “if the
world is not the work of chance, it must have had an intelligent maker;”
“either mankind are capable of rising into civilization unassisted or the first
beginning of civilization must have come from above.” (Whately 1825,
89)

The distinction between the categorical and the hypothetical merely taxonomic, in that
there is no substantive semantic or logical theory of the hypothetical propositions given
at this stage. Negation tends to be subsumed into a predicate feature and placed in the
Aristotelian taxonomy, and conjunction is largely ignored.

After the theory of proposition, a theory of syllogistic validity is given, in varying
levels of detail. (From this point, textbooks often branch out to less core logical topics,
such as inductive reasoning or rhetorical techniques.)

Three noteworthy features of the treatment of connectives then emerge from this
standard program:

1. Core Conditionality: As noted above, conjunction rarely receives overt atten-
tion from the modern-era logicians, and negation tends not to be treated as a
sentential connective, but rather as a term modifier. The category of molecular proposition that is most frequently recognized is that of the conditional, or hypothetical, proposition. Disjunctions tend to be subsumed to the category of hypotheticals, either by identification as a special case (as in the quotation from Whately above), or via an explicit reduction to conditionals:

In *Disjunctive* propositions two or more assertions are so connected by a disjunctive particle, that one (and but one of them) must be true, as *Either riches produce happiness, or Avarice is most unreasonable*. *Either it is day, or it is night.* – This kind of propositions are easily resolvable into *Hypothetical*; as, *If Riches do not produce happiness, then, Avarice is most unreasonable.* (Benthem 1773, 47)

2. **Inferential Categorical Reductionism**: Since among modern-era logicians there is rarely any explicit semantic analysis given to molecular propositions, the analysis of them tends to come via remarks on how their inferential potential can be analyzed into that of categorical propositions whose inferential roles are already specified through the theory of Aristotelian syllogistics.

The most common technique is to subsume both modus ponens and transitive chaining of conditionals to “Barbara” syllogisms:

We must consider every conditional proposition as a universal-affirmative categorical proposition, of which the terms of entire propositions. …To say, “if Louis is a good king, France is likely to prosper,” is equivalent to saying, “The case of Louis being a good king, is a case of France being likely to prosper:” and if it be granted as a minor premiss to the conditional syllogism, that “Louis is a good king;” that is equivalent to saying, “the present case is the case of Louis being a good king;” from which you will draw a conclusion in Barbara (viz. “the present case is a case of France being likely to prosper,”) exactly equivalent to the original conclusion of the conditional syllogism. (Whately 1825, 136)

Additionally, modus tollens inferences can in the same manner be subsumed to “Celarent” syllogisms.

3. **Content/Force Ambivalence**: Modern-era logicians tend to describe connectives in a way that is compatible both with the contemporary view that these connectives are content-level operators and also with the view that connectives are indicators of speech-act force. This tendency is most prominent in the case of negation. It is standard to speak of negation as creating *denials*, in a way that leaves unclear whether denials are a particular type of content to be asserted, or are an alternative speech act to assertion:

Because Benthem is reading the disjunction as exclusive, he presumably takes the English “if…then” construction to be biconditional in force, to secure the stated equivalence.
Affirmation and Denial belong to the very nature of things; and the distinction, instead of being concealed or disguised to make an imaginary unity, should receive the utmost prominence that the forms of language can bestow. Thus, besides being either universal or particular in quantity, a proposition is either affirmative or negative. (Bain 1874, 83-84)

Modality also shows this ambivalence – necessity, for example, is often described as a way of asserting or predicating, rather than as an aspect of the content asserted:

The modality of judgments is a quite peculiar function. Its distinguishing characteristic is that it contributes nothing to the content of the judgment (for, besides quantity, quality, and relation, there is nothing that constitutes the content of a judgment), but concerns only the value of the copula in relation to thought in general. (Kant 106)

In general, the modern era contrasts sharply with the ancient and medieval periods in its radically diminished interest in the logic of modality.

At times, even the quantificational forces of universality and existentiality are characterized in speech-act/force terms:

A Modal proposition may be stated as a pure one, by attaching the mode to one of the Terms: and the Proposition will in all respects fall under the foregoing rules. . . . E.g., “man is necessarily mortal:” is the same as “all men are mortal.” . . . Indeed every sign (or universality or particularity) may be considered as a Mode. (Whately 1825, 108)

8 Bolzano

Bernard Bolzano (1781-1840) raises the analysis of logical consequence to a new level of rigor and detail. Bolzano’s account of logical consequence begins with an account of propositional structure. Bolzano shares the standard picture of modern-era logic, according to which the paradigm case of a proposition is a subject-predicate proposition. In particular, Bolzano takes the form A has b, for terms A and b, to be the common underlying structure of all propositions:

The concept of having, or still more specifically, the concept signified by the word has, is present in all propositions, which are connected with each other by hat has in a way indicated by the expression: A has b. One of these components, namely the one indicated by A, stands as if it was supposed to represent the object with which the proposition is concerned. The other, b, stands as if it was supposed to represent the property the proposition ascribes to that object. (§127)

Again in keeping with the standard modern-era picture, Bolzano is committed to this single subject-predicate form subsuming molecular propositions:
But there are propositions for which it is still less obvious how they are supposed to fall under the form, $A$ has $b$. Among these are so-called hypothetical propositions of the form, if $A$ is, then $B$ is, and also disjunctive propositions of the form, either $A$ or $B$ or $C$, etc. I shall consider all of these propositional forms in greater detail in what follows, and it is to be hoped that it will then become clear to the reader that there are no exceptions to my rule in these cases. ($§$127)

However, Bolzano’s conception of the nature of the propositional components $A$ and $b$ is particularly expansive, which leads him in an interesting direction. Bolzano anticipates Frege’s view that propositions are timelessly true:

But it is undeniable that any given proposition can be only one of the two, and permanently so, either true and then true for ever, or false, and, again, false forever. ($§$140)

At the same time, he recognizes that ordinary practice involves ascribing changing truth values to propositions:

So we say that the proposition, wine costs 10 thalers the pitcher, is true at this place and this time, but false at another place or another time. Also that the proposition, this flower has a pleasant smell, is true or false, depending on whether we are using the this in reference to a rose or a carrion-flower, and so on. ($§$140)

Bolzano accounts for the ordinary practice through the idea that “it is not actually one and the same proposition that reveals this diversified relationship to the truth”, but rather that “we are considering several propositions, which only have the distinguishing feature of arising out of the same given sentence, in that we regard certain parts of it as variable, and substitute for them sometimes this idea and sometimes that one.” ($§$140)

Once accounting for these broadly contextual features has introduced the idea of treating certain propositional components as variable, that idea then finds profitable application in accounting for logical consequence. Bolzano contrasts the two sentences:

1. The man Caius is mortal.
2. The man Caius is omniscient.

When we “look on the idea of Caius… as one that is arbitrarily variable”, we discover that (1) is true for many other ideas (the idea of Sempronius, the idea of Titus), but also false for many other ideas (the idea of a rose, the idea of a triangle – because these ideas do not truthfully combine with the idea of a man). On the other hand, (2) is false regardless of what idea is substituted for the idea of Caius.

Sometimes when we pick a proposition $P$ and a collection of constituents $i_1, \ldots, i_n$ of that proposition, we will discover that any substitutions for those constituents will result in a true proposition. In such a case, Bolzano says that $P$ is valid with respect to $i_1, \ldots, i_n$. A proposition that is valid with respect to any collection of constituents is then called analytic. The resulting conception of analyticity is not a perfect match for contemporary notions of analyticity, as Bolzano’s own example makes clear:
(3) A morally evil man deserves no respect. (§148) (Valid with respect to the constituent man.)

More generally, a proposition is analytic in Bolzano’s sense just in case it is a true sentence with a wide-scoped non-vacuous universal second-order quantifier. Many such sentences will be intuitively non-analytic, in that they will depend for their truth on substantive contingencies about the world, as in:

(4) ∀P (every senator who owns a P is within 1,000,000 miles of Washington D.C.)

Bolzano then defines a sequence of relations between propositions using the tools of propositional constituent variability:

1. **Compatibility, Incompatibility**: A collection of propositions $P_1, \ldots, P_n$ is compatible with respect to a collection of constituents $i_1, \ldots, i_m$ if and only if there are substitution instances for $i_1, \ldots, i_m$ such that $P_1, \ldots, P_n$ are all true under those substitutions. **Incompatibility** is then lack of compatibility.

2. **Derivability, Equivalence, Subordination**: One set $C_1, \ldots, C_n$ of propositions is derivable from another set $P_1, \ldots, P_m$ of propositions with respect to a collection of constituents $i_1, \ldots, i_k$ if and only if $C_1, \ldots, C_n$ are compatible with $P_1, \ldots, P_m$ with respect to $i_1, \ldots, i_k$, and every substitution instance for $i_1, \ldots, i_k$ which makes true all of $P_1, \ldots, P_m$ also makes true all of $C_1, \ldots, C_n$. **Equivalence** is then mutual derivability; **subordination** is derivability without equivalence (with premises subordinate to conclusion).

3. **Concatenation**: A collection of propositions $A_1, \ldots, A_n$ is concatenating with (or intersecting with, or independent of) a collection of propositions $B_1, \ldots, B_m$ with respect to a collection of constituents $i_1, \ldots, i_k$ if neither collection is derivable from the other with respect to $i_1, \ldots, i_k$, but the entire collection $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ is compatible with respect to $i_1, \ldots, i_k$.

For each of these relations, its holding is equivalent in an obvious way to the truth of a second-order quantified sentence – the compatibility of $P_1, \ldots, P_n$ with respect to $i_1, \ldots, i_m$ is equivalent to the truth of the $m$-fold second-order existential quantification of the conjunction of $P_1$ through $P_n$ with the propositional constituents $i_1$ through $i_m$ replaced with second-order variables, for example. Bolzano consistently leaves these propositional relations relativized to a choice of collection of propositional constituents – he does not have a notion of “derivability simpliciter”, but rather a family of derivability notions with no preferred or canonical member. However, if the object language is thought of as a first-order language and derivability is relativized to the collection of standard logical constants, the resulting notion of derivability almost matches the Tarskian model-theoretic definition, except that Bolzano places an additional constraint that premises and conclusion be consistent – thereby ruling out derivations from contradictory premises.

Relationships such as compatibility, derivability, and concatenating with would, in a contemporary setting, be taken as relations in the metalanguage. However, Bolzano makes no object language/metalanguage distinction, and is, for example, explicit that
“we make use of [if and then] to express the relationship of derivability of one proposition from one or more other propositions.” (§179) But Bolzano does not simply equate derivability and conditionality. He considers the example:

(5) If Caius remains silent on this occasion, he is ungrateful.

and rejects the view that:

…there is a certain idea (say the idea of Caius) in those propositions, which can be regarded as variable with the result that every assignment of it which makes the first proposition true also would make the second proposition true. (§179)

Instead, Bolzano’s picture is that:

What [(5)] means to say is that among Caius’ circumstances there are some such that they are subject to the general principle that anyone who remains silent under such circumstances is ungrateful. …For example, if Caius is dead, then Sempronius is a beggar. With these words, too, all we are saying is that there are certain relationships between Caius and Sempronius of which the general principle holds true that of any two men of whom one (in Caius’ circumstances) dies the other (in Sempronius’ circumstances) is reduced to beggarhood. (§179)

Bolzano thus gives a form of premise semantics for conditionals. A conditional of the form if A, then B is true just in case there are true supplementary principles which together with A entail B. Bolzano appears to require that these supplementary principles from the combination of which with A, B can be derived.

Because of the close relationship between derivability and conditionality, a sequence of observations Bolzano makes regarding the logic of derivability entail observations regarding the logic of the conditional. Bolzano is thus committed to the following inferences for conditionals:

1. \((A \land \top) \rightarrow B \Rightarrow A \rightarrow B\)
2. \(A \rightarrow B, \neg A \Rightarrow \neg B\)
3. \(\neg \exists p(\neg p \land (A \rightarrow p)) \Rightarrow A\)
4. \(\Rightarrow \neg(A \rightarrow \neg A)\) [Even for contradictory A, given Bolzano’s compatibility requirement on derivability.]
5. \((A \rightarrow B), (\neg A \rightarrow B) \Rightarrow B \equiv \top\)
6. \((A \land B) \rightarrow C \Rightarrow (\neg C \land B) \rightarrow \neg A\)
7. \(((A \land B) \rightarrow C), ((A \land \neg B) \rightarrow C) \Rightarrow A \rightarrow C\)
8. \(A \rightarrow B, C \rightarrow D \Rightarrow (A \land C) \rightarrow (B \land D)\)

\(\top\) here is an arbitrary tautology.
9. $A \to B, (B \land C) \to D \Rightarrow (A \land C) \to D$

Bolzano anticipates Lewis Carroll 1895’s regress argument against conflating inference rules with conditionalized premises. In addition to the relation of derivability, Bolzano sets out a hyperintensional relation of ground and consequence, which holds between two propositions when the first (partially) grounds or establishes or underlies the truth of the latter. The ground and consequence relation is thus, unlike the derivability relation, asymmetric. Bolzano then considers the ground-consequence relation between the grounds:

(6) Socrates was an Athenian.
(7) Socrates was a philosopher.

and the consequence:

(8) Socrates was an Athenian and a philosopher.

The crucial question is whether (6) and (7) form the complete grounds for (8), by virtue of the derivability relation among them, or whether to the complete grounds needs to be added a sentence corresponding to the inference rule:

- $A, B \Rightarrow A \land B$

The relevant sentence would then be:

(9) If proposition $A$ is true and proposition $B$ is true, proposition $A \land B$ is true.

Bolzano then objects that in adding this sentence as an addition (partial) ground for (8):

- One is really claiming that propositions $M, N, O, \ldots$ are only true because this inference rule is correct and propositions $A, B, C, D, \ldots$ are true. In fact, one is constructing the following inference:

  - If propositions $A, B, C, D, \ldots$ are true, then propositions $M, N, O, \ldots$ are also true; now propositions $A, B, C, D, \ldots$ are true; therefore propositions $M, N, O, \ldots$ are also true.

  Just as every inference has its inference rule, so does this one. …Now if one required to start with that the rule of derivation be counted in the complete ground of truths $M, N, O, \ldots$, along with truths $A, B, C, D, \ldots$, the same consideration forces one to require that the second inference rule … also be counter in with that ground. …One can see for oneself that this sort of reasoning could be continued ad infinitum …But this would seem absurd. (§199)

60 years before Lewis Carroll, Bolzano also sees that a global demand that inferential principles be articulated and deployed as premises will lead to a regress which is incompatible with the presumed well-founded of (in this case) the grounding relation. While a separation between using a rule and stating a rule need not track with a distinction between derivability and conditionality, Bolzano’s sensitivity to the threatened paradox here does suggest some caution on his part regarding a too-tight correlation between the latter two.
Bolzano’s propositional relations of compatibility, derivability, and concatenation depend on an underlying notion of truth relative to an index, in the form of a quantificationally-bound choice of values for propositional components construed as variable. As such, these relations are intensional relations. Bolzano then recognizes that it is possible to distinguish from these intensional propositional relations a class of extensional relations:

There was no talk of whether the given propositions were true or false so far as the relations among propositions considered [earlier] were concerned. All that was considered was what kind of a relation they maintained, disregarding truth or falsity, when certain ideas considered variable in them were replaced by any other ideas one pleased. But it is as plain as day that it is of the greatest importance in discovering new truths to know whether and how many true – or false propositions there are in a certain set. (§160)

Extensional propositional relations of this sort are a natural home for connectives of conjunction and disjunction. Because Bolzano is happy to talk of a set of propositions being true (and because he does not consider embedded contexts), he has little need for a notion of conjunction. He does, however, set out a number of types of disjunction.

1. **Complementarity**: A collection of propositions is complementary or auxiliary if at least one member of it is true.

2. **One-Membered Complementarity**: A collection of propositions is one-membered complementary if exactly one member of it is true. Bolzano says that this relation “is ordinarily called a disjunction” (§160)

3. **Complex Complementarity**: A collection of propositions is complexly or redundantly complementary if two or more members of it are true.

Bolzano’s range of disjunctions are thus explicitly truth-functional, and include both inclusive disjunction (complementarity) and exclusive disjunction (one-membered complementarity), as well as the novel complex complementarity.

Bolzano also recognizes intensionalized versions of disjunction – what he calls “a formal auxiliary relation” – in which “this relationship of complementarity can subsist among given propositions \( M, N, O, \ldots \). no matter what ideas we substitute for certain ideas \( i, j, \ldots \) regarded as variable in them.” (§160) Using intensionalized disjunction, Bolzano can then define two further varieties of disjunction which cannot be non-trivially specified for extensional disjunction:

4. **Exact Complementarity**: Propositions \( P_1, \ldots, P_n \) are exactly complementary with respect to propositional constituents \( i_1, \ldots, i_m \) if and only if both (a) for any choice of values for \( i_1, \ldots, i_m \), at least one member of \( P_1, \ldots, P_n \) is true, and (b) there is no proper subset of \( P_1, \ldots, P_n \) for which condition (a) also holds.

5. **Conditional Complementarity**: Propositions \( P_1, \ldots, P_n \) are conditionally complementary with respect to propositions \( A_1, \ldots, A_m \) and propositional constituents \( i_1, \ldots, i_k \) if and only if for any choice of values for \( i_1, \ldots, i_k \) which makes true all of \( A_1, \ldots, A_m \), at least one member of \( P_1, \ldots, P_n \) is true.
Bolzano characterizes disjunction as a generalization of the subcontrariety relation from the square of opposition. In this context, he notes some preconditions on successful subcontrary relations. “Some A are B” and “Some A are not B” are subcontraries only conditional on the assumption that A is a denoting concept. Similarly, “If A then B” and “If Neg. A then B” are subcontraries only on the assumption that B is true. Bolzano thus rejects a principle of antecedent conditional excluded middle:

\[ \neg (A \rightarrow B) \Rightarrow \neg A \rightarrow B \]

A commitment to the falsity of a conditional with true antecedent and false consequent – or, equivalently, a commitment to the validity of modus ponens – is sufficient to reject antecedent conditional excluded middle.

Bolzano’s remarks on negation are relatively cursory. He shares with the standard modern-era logic a tendency to equivocate between a content-level and a speech-act-level analysis of negation. He frequently refers to “Neg. A” as the “denial” of A in a way that suggests that negation is a speech-act marker. But he also clearly sets out the truth-functional analysis of negation via connections between negation and falsity:

Where A designates a proposition I shall frequently designate its denial, or the proposition that A is false, by “Neg. A” (§141)

Bolzano regularly appeals to the truth-functional analysis of negation when arguing for metatheoretic results, as when he argues that compatibility of A, B, C, D, … need not be preserved under negation of all elements:

For it could well be that one of the propositions … e.g. A, would not only be made true by some of the ideas that also made all the rest B, C, D, … true, but besides that would be made true by all of the ideas by which one of those propositions, e.g. B, is made false and consequently the proposition Neg. B made true. In this case there would be no ideas that made both propositions, Neg. A and Neg. B, true at the same time. (§154)

Bolzano recognizes that equivalence of propositions is preserved under negation (§156), and is explicit that propositions are equivalent to their own double negation:

I cite the example of ‘something’, which is equivalent to the double negative concept, ‘not not something’, and to every similar concept containing an even number of negations. (§96)

9 Boole

George Boole (1779-1848) is one of the most radical innovators in the history of logic. He differs sharply from his predecessors in the representational framework he uses for representing logical inferences, in the scope of logical arguments he seeks to analyze, and most markedly in the tools he uses for determining the validity and invalidity of inferences. Boole’s approach to logic is heavily algebraic, in that he analyzes and solves logical problems by representing those problems as algebraic equations, and then laying a system of permissible algebraic manipulations by which the resulting
equations can be reworked into forms that reveal logical consequences. While Boole thus shares with Leibniz a focus on algebraic approaches to logic, Boole’s system is considerably more developed and complex than Leibniz’s.

Boole’s core algebraic representational system consists of four central elements:

1. **Variables**: Boole’s logical system admits of two interpretations, the *primary* interpretation and the *secondary* interpretation. On the primary interpretation, variables represent classes. On the secondary interpretation, variables represent propositions. Boole is thus able to model monadic quantificational logic with the primary interpretation and propositional logic with the secondary interpretation.

   - Boole distinguishes between primary propositions, which are propositions which directly represent the state of the world, and secondary propositions, which are “those which concern or relate to Propositions considered as true or false.” (Boole 1854, 160). As a result, Boole’s characterization of molecular propositions is metalinguistic and truth-involving:

     Secondary propositions also include all judgments by which we express a relation or dependence among propositions. To this class or division we may refer conditional propositions, as “If the sun shine the day will be fair.” Also most disjunctive propositions, as, “Either the sun will shine, or the enterprise will be postponed.” …In the latter we express a relation between the two Propositions, “The sun will shine,” “The enterprise will be postponed,” implying that the truth of the one excludes the truth of the other.’

     (Boole 1854, 160)

Boole also holds that secondary propositions can, in effect, be reduced to primary propositions (and hence the secondary interpretation of the logical algebra to the primary interpretation) by equating each primary proposition with the class of times at which it is true, and then construing secondary propositions as claims about these classes. Thus, for example:

   Let us take, as an instance for examination, the conditional proposition, “If the proposition *X* is true, the proposition *Y* is true. An undoubted meaning of this proposition is, that the *time* in which the proposition *X* is true, is *time* in which the proposition *Y* is true. (Boole 1854, 163)

Boole’s conception of the conditional is in effect a strict conditional analysis – however, rather than requiring appropriate truth value coordination between antecedent and consequent with respect to every *world*, Boole requires it with respect to every *time*. Boole’s willingness to represent molecular claims by classes of truth-supporting indices represents an early form of “possible-worlds” semantics, again with times replacing worlds.

The symbol $v$ is reserved by Boole as the “indefinite” symbol, the role of which is set out below.

2. **Constants**: The constants 0 and 1 play a central role in Boole’s system, representing on the primary interpretation the empty class and the universal class, and
on the secondary interpretation falsity and truth. Other numerical constants, as we will see below, occasionally play a role in the computational mechanics of his system; these other constants lack a straightforward interpretation.

3. Operations: Boole uses a number of arithmetic operations on terms. The two central operations in his system are addition (‘+’) and multiplication (‘×’). Multiplication, under the primary interpretation, represents simultaneous application of class terms:

If \( x \) alone stands for “white things”, and \( y \) for “sheep”, let \( xy \) stand for “white sheep”. (Boole 1854, 28)

Multiplication thus acts as an intersection operator on classes. Boole’s description of the role of multiplication under the secondary interpretation is somewhat baroque:

Let us further represent by \( xy \) the performance in succession of the two operations represented by \( y \) and \( x \); i.e. the whole mental operation which consists of the following elements, viz., 1st, The mental selection of that portion of time for which the proposition \( Y \) is true, 2ndly, The mental selection, of that portion of time, of such portion as it contains of the time in which the proposition \( X \) is true, – the result of these successive processes being the fixing of the mental regard upon the whole of that portion of time for which the propositions of \( X \) and \( Y \) are true. (Boole 1854, 165)

Once the excess psychologism is stripped away, however, this amounts to a treatment of multiplication as standard conjunction-as-intersection in an intensional setting. Under the primary interpretation, Boole takes addition to be the operation of “forming the aggregate conception of a group of objects consisting of partial groups, each of which is separately named or described” (Boole 1854, 32). This is roughly an operation of set union, but it acts under a presupposition of disjointness of the two sets. Thus when Boole wants to specify a straightforward union, he needs to add some slight epicycles:

According to the meaning implied, the expression, “Things which are either \( x \)’s or \( y \)’s,” will have two different symbolic equivalents. If we mean, “Things which are \( x \)’s, but not \( y \)’s, or \( y \)’s, but not \( x \)’s,” the expression will be \( x(1 - y) + y(1 - x) \); the symbol \( x \) standing for \( x \)’s, \( y \) for \( y \)’s. If, however, we mean, “Things which are either \( x \)’s, or, if not \( x \)’s, then \( y \)’s,” the expression will be \( x + y(1 - x) \). This expression supposes the admissibility of things which are both \( x \)’s and \( y \)’s at the same time. (Boole 1854, 56)

Boole thus distinguishes between disjoint and non-disjoint union, but the expression \( x + y \) itself corresponds to neither. On the secondary interpretation, addition similarly corresponds to disjunction with a presupposition of the impossibility of the conjunction.
Multiplication and addition thus serve roughly as meet and join operations on a boolean algebra of classes. But the contemporary algebraic characterization is only an imperfect fit for Boole’s actual system – the presupposition of disjointness on addition, the additional arithmetic operations detailed below, and the actual computational mechanisms of Boole’s system all distinguish it from contemporary algebraic semantics.

Boole also makes use of inverse operations of subtraction and division to his core operations of addition and multiplication, as well as occasional use of exponentiation. Subtraction is interpreted as “except”, with \( x - y \) representing the class \( x \), except those member of \( x \) which are also \( y \). The interpretation of division is not taken up by Boole. Boole then, based on the arithmetic equivalence between \( x - y \) and \( -y + x \), allows himself use of a unary operation of negation, which again receives no explicit interpretation.

4. **Equality**: The arithmetic operations recursively combine variables and constants to create complex terms. The mathematical familiarity with recursively embedded terms creates in Boole a sensitivity to embedded occurrences of sentential connectives that is unusual in the modern-era logicians, as in:

\[
\text{Suppose that we have } y = xz + vx(1 - z) \text{ Here it is implied that the time for which the proposition } Y \text{ is true consists of all the time for which } X \text{ and } Z \text{ are together true, together with an indefinite portion of the time for which } X \text{ is true and } Z \text{ is false. From this it may been seen, 1st, That if } Y \text{ is true, either } X \text{ or } Z \text{ are true together, or } X \text{ is true and } Z \text{ false;2ndly, If } X \text{ and } Z \text{ are true together, } Y \text{ is true. (Boole 154, 173)}
\]

Terms, whether simple or complex, are then combined into full expressions with the only predicative element of Boole’s system – the identity sign. All full sentences in Boole’s algebraic representation are equations, and Boole’s symbolic manipulation methods are all techniques for manipulating equations.

The centerpiece of Boole’s algebraic manipulation methods is the validity \( x^2 = x \). This validity represents for Boole the triviality that the combination of any term with itself picks out the same class as the term itself. \( x^2 = x \) marks the primary point of departure of Boole’s logical algebra from standard numerical algebra – since from \( x^2 = x \) we can derive \( x^2 - x = 0 \) or \( x(x - 1) = 0 \), we obtain \( x = 0 \) and \( x = 1 \) as the two solutions to the distinctive validity, and establish 0 and 1 as terms of special interest in the logical algebra. For Boole, this special interest manifests in two ways. First, it establishes 0 and 1 as boundary points for the representational system, underwriting their roles as empty class/falsity and universal class/truth. Second, it allows him to appeal to any computational method which is valid for the special cases of 0 and 1, even if it is not globally arithmetically valid.

In deriving logical consequences, Boole uses three central algebraic techniques: **reduction**, for combining multiple premises into a single equation, **elimination**, for removing unwanted variables from equations, and **development**, for putting equations into a canonical form from which conclusions can easily be read off.
1. **Reduction:** Suppose we have two separate equations, such as \( x = y \) and \( xy = 2x \), and we wish to combine the information represented by these two equations. We first put each equation into a canonical form by setting the right-hand side to 0:

- \( x - y = 0 \Rightarrow x + (-y) = 0 \)
- \( xy - 2x = 0 \Rightarrow xy + (-2x) = 0 \)

Since, when we limit our field to 0 and 1, the only way a sum can be 0 is if each term is 0, Boole takes the informational content of each equation to be that each term is equal to 0. The two equations can thus be combined by adding, so long as there is no undue cancellation of terms thereby. When equations are in the properly developed form, as explained below, squaring an equation will have the effect of squaring the constant coefficient of each term, ensuring that all coefficients will be positive and thus that there will be no undue cancellation. Thus Boole’s procedure of reduction is to put each equation into canonical form, square each equation, and then add all the squared equations to produce a single equation.

2. **Elimination:** At times, we may be uninterested in information regarding some of the terms in an equation. This is particularly likely in cases involving Boole’s indefinite term \( v \). Because Boole’s sole predicate is equality, he has no direct means for representing subset/containment relations between classes. To indicate that \( x \) is a subclass of \( y \) Boole thus uses the equation \( x = vy \), where the term \( v \) represents “a class indefinite in every respect” (Boole 1854, 61). Thus the universal generalization “All \( X \) are \( Y \)” is represented as \( x = vy \), as is (on the secondary interpretation) the conditional “If \( X \), then \( Y \)”. But information about the wholly indefinite class \( v \) will be unwanted in the final analysis, so we need a technique for eliminating it from an equation.

Boole’s method of eliminating a variable from an equation proceeds by taking an equation \( f(x_1, \ldots, x_n) = 0 \) in standard form. A particular variable \( x_j \) is targeted for elimination, and a new equation is formed by setting equal to 0 the product of the left-hand term with \( x_j \) replaced by 0 with the left-hand term with \( x_j \) replaced by 1:

- \( f(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n) \times f(x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n) = 0 \)

The resulting equation is free of \( x_j \).

Boole justifies the method of elimination via an intricate sequence of algebraic maneuvers, the logical significance of which is far from clear (Boole 1854, 101-102). From a contemporary point of view, the crucial fact is that in a boolean algebra, \( f(x) = f(1)x + f(0)(-x) \).

3. **Development:** Boole sets out a canonical form into which equations are to be placed to allow informational conclusions to be read off of them. When an equation contains only a single variable \( x \), it is in canonical form when it is of the form \( ax + b(1 - x) = 0 \) – at this point, it gives a characterization both of the class
x, and of the complement of the class x, which are the two classes made available by the single class term x.

To determine the coefficients a and b, we set \( a = f(1) \) and \( b = f(0) \). Here again, Boole’s argument for the value of the coefficient is involved and algebraic (and a second argument is given using Taylor/Maclaurin series). In contemporary terms, the development coefficients are a reflex of the underlying truth-functionality of the logic, which allows propositions to be fully characterized by their behavior when sentence letters are set to true and when sentence letters are set to false.

In the general case, an equation in \( n \) variables \( x_1, \ldots, x_n \) is in canonical form when it is in the form:

\[
\bullet \quad a_1 x_1 \ldots x_n + a_2 (1 - x_1) x_2 \ldots x_n + \ldots + a_2^n (1 - x_1) \ldots (1 - x_n)
\]

Here each of the \( 2^n \) classes that can be produced via intersections of the classes \( x_1 \) to \( x_n \) and their complements is individually characterized. The development coefficients are then of the form \( f(\epsilon_0, \ldots, \epsilon_n) \), where \( \epsilon_j = 0 \) if the term contains \( 1 - x_j \), and 1 if the term contains \( x_j \).

Once in canonical form, conclusions can be read off using the nature of the coefficients, via an intricate set of rules that Boole sets out.

Consider now an example worked out in full detail. Let the two premises be “If Socrates is a man, he is an animal”, and “If Socrates is an animal, he is mortal”. Let \( x \) be the class of times at which Socrates is a man, \( y \) be the class of times at which Socrates is an animal, and \( z \) be the class of times at which Socrates is mortal. Then we begin with the two equations:

\[
\bullet \quad x = vy \\
\bullet \quad y = vz
\]

First we put each equation into standard form, and eliminate \( v \) from each:

\[
\bullet \quad x = vy \Rightarrow x - vy = 0 \Rightarrow (x - 1y)(x - 0y) = 0 \Rightarrow (x - y)x = 0 \Rightarrow x^2 - xy = 0 \Rightarrow x - xy = 0
\]

[The last step follows from Boole’s fundamental principle that \( x^2 = x \), which more generally allows elimination of exponents.]

\[
\bullet \quad y = vz \Rightarrow y - vz = 0 \Rightarrow (y - 1z)(y - 0z) = 0 \Rightarrow (y - z)y = 0 \Rightarrow y^2 - yz = 0 \Rightarrow y - yz = 0
\]

We then square both equations in preparation for reduction:

\[
\bullet \quad (x - xy)^2 = 0 \Rightarrow x^2 - 2x^2y + x^2y^2 = 0 \Rightarrow x - 2xy + xy = 0 \Rightarrow x - xy = 0
\]

\[
\bullet \quad (y - yz)^2 = 0 \Rightarrow y^2 - 2yz^2 + y^2z^2 = 0 \Rightarrow y - 2yz + yz = 0 \Rightarrow y - yz = 0
\]

\[\text{Note that the elimination of } v \text{ avoids the need to confront the delicate issue of whether the two occurrences of } v \text{ represent the same class.}\]
Adding the two equations together yields:

- \( x - xy + y - yz = 0 \)

Solving for \( x \) yields:

- \( \frac{y(1-z)}{1-y} \)

We then develop the right-hand side in \( y \) and \( z \):

- \( x = \frac{0}{v}yz + \frac{1}{v}y(1 - z) + 0(1 - y)z + 0(1 - y)(1 - z) \)

The coefficient \( \frac{0}{v} \) indicates that \( x \) is to be set equal to an indefinite portion of that term, and the coefficient \( \frac{1}{v} \) indicates that that term is to be set independently equal to zero. The other two terms, with coefficients of 0, disappear and can be disregarded. We thus obtain two results:

1. \( y(1 - z) = 0 \). This equation shows that there is no time at which Socrates is an animal but not mortal, which simply reconfirms the second conditional.

2. \( x = vyz \). Since \( y = vz \), we have by substitution \( x = v^2z^2 = vz \). This equation is the representation of the conditional “If Socrates is a man, he is mortal”. The transitive chaining of conditionals is thus derived.

Boole’s algebraic system gives him the ability to represent all of the standard truth-functional connectives, and capture their truth-functionally determined inferential properties. His conditional, in particular, is inferentially equivalent to a material conditional. There are, however, some representational peculiarities that derive from his algebraic approach. Conjunctions and disjunctions are represented as terms – \( xy \) is the conjunction of \( x \) and \( y \), \( x(1 - y) + y(1 - x) \) the exclusive disjunction of \( x \) and \( y \), and \( x + (1 - x)y \) the inclusive disjunction of \( x \) and \( y \). Conditionals, on the other hand, are represented by equations. The conditional whose antecedent is \( x \) and whose consequent is \( y \) is represented by the equation \( x = vy \). This difference is driven by the fact that Boole’s algebraic approach means that biconditional claims are really at the foundation of his expressive resources. Boole can, for example, straightforwardly express the biconditional between “\( x \) and \( y \)” and “\( y \) and \( x \)” using the equation \( xy = yx \). Conditionals come out as equations only because of the peculiar role of the indefinite class \( v \), which allows conditional relations to be expressed with biconditional resources.

Boole’s work in logic was tremendously influential, first in the United Kingdom and later more broadly in Europe and America. British logicians such as Augustus De-Morgan (1806-1871), William Jevons (1835-1882), and John Venn (1834-1923) took up Boole’s algebraic approach, and made steps toward simplifying and improving it. The followers of Boole clearly see themselves as enfants terribles working against a retrograde logical tradition, as can be seen as late as 1881 in the introduction to Venn’s Symbolic Logic:

There is so certain to be some prejudice on the part of those logicians who may without offence be designated as anti-mathematical, against any work professing to be a work on Logic, in which free use is made of the
symbols $+$ and $-$, $\times$ and $\div$, (I might almost say, in which $x$ and $y$ occur in place of the customary $X$ and $Y$) that some words of preliminary explanation and justification seem fairly called for. Such persons will without much hesitation pronounce a work which makes large use of symbols of this description to be mathematical and not logical. (Venn 1881, 1)

Much of the focus of the later Booleans is thus on improving the interpretability of Boole’s formalism, so that the intermediate computational steps have clearer meaning. As early as the 1860s, DeMorgan and Jevons suggest that the presupposition of disjointness be dropped from the interpretation of $+$, making it into a simple operation of class union. Here the discussion focuses on the status of the law $x + x = x$, which Boole rejects but DeMorgan and Jevons endorse. (This law is difficult for Boole to accept, because unlike his preferred $x^2 = x$, it does not have 0 and 1 as solutions, but only 0.) DeMorgan introduces, and Jevon picks up on, a notational convention in which Boole’s use of the subtraction symbol as a monadic class negation operation is replaced by the use of capital and lower-case letters to represent a class and its negation.

And Venn famously shows how many of Boole’s logical conclusions can be reached diagrammatically with what came to be called Venn diagrams. There results a small cottage industry in specifying techniques for producing Venn diagrams for four or more sets. Lewis Carroll, in Carroll 1896, shows how overlapping rectangles can produce an elegant four-set diagram:

![Venn diagram](image)

Venn 1881 sets out a clever representational trick in which a specified region represents non-membership in a given set, in order to produce a useable five-set diagram:
Here the smaller central oval represents objects which are not members of the larger central oval.

10 Frege

Gottlob Frege (1848-1925) brings logic to its contemporary form, albeit in the dress of a rather forbidding notation. In his 1879 *Begriffsschrift*, he sets out the syntax, semantics, and proof theory of first-order quantified logic with a level of detail and precision unmatched in the work of his predecessors and contemporaries. Frege gives a full recursive syntax for his logical language. A primitive stock of terms for expressing “judgeable contents” are first combined with a horizontal or content stroke, which “binds the symbols that follow it into a whole” and a vertical or judgment stroke, which indicates that the stated content is asserted, rather than merely put forth to “arouse in the reader the idea of” that content. (§2) Thus Frege begins with symbolic representations such as:

- \( \vdash A \)

Four further components are then added to the syntax:

1. **Negation**: The placement of a “small vertical stroke attached to the underside of the content stroke” expresses negation:

\[ \neg A \]

---

The most forbidding aspect of Frege’s notation is his extensive use of two-dimensional symbolic representations, as seen below. The two-dimensionality has the immediate benefit of readily representing the hierarchical syntactic structure of logical expressions. In his “On Mr Peano’s Conceptual Notation and My Own” (1897), Frege recognizes that a linear notation can also represent the hierarchical relations, but holds that his two-dimensional notation continues to have an advantage in ease of use:

Because of the two-dimensional expanse of the writing surface, a multitude of dispositions of the written signs with respect to one another is possible, and this can be exploited for the purpose of expressing thoughts. … For physiological reasons it is more difficult with a long line to take it in at a glance and apprehend its articulation, than it is with shorter lines (disposed one beneath the other) obtained by fragmenting the longer one – provided that this partition corresponds to the articulation of the sense. (Frege 1897, 236)

Subsequent history has not unambiguously validated this view of Frege’s.
If we think of the negation stroke as producing a content stroke both before and after it, we have a natural recursive mechanism that allows for embedding of negations:

\[ \neg A \]

2. **Conditional**: Frege expresses conditionals using a two-dimensional branching syntax. Thus:

\[ \begin{array}{c}
\vdash A \\
\vdash B
\end{array} \]

expresses the conditional \( A \rightarrow B \). Again, the syntax allows recursive embedding, and also combination with negation.

\[ \begin{array}{c}
(A \rightarrow B) \rightarrow (C \rightarrow D) : \\
D \\
C \\
B \\
A
\end{array} \]

3. **Generality**: Frege expresses universal quantification using a concavity in the content stroke:

\[ \vdash F(x) \]

The recursive specification of the syntax then allows Frege to distinguish scopes for quantifiers, so that he can, for example, distinguish:

\[ \begin{array}{c}
\forall x(F(x) \rightarrow G(x)) : \\
G(x) \\
F(x)
\end{array} \]

\[ \begin{array}{c}
\forall xF(x) \rightarrow \forall xG(x) : \\
G(x) \\
F(x)
\end{array} \]

or place one quantifier within the scope of another:

\[ \begin{array}{c}
\forall x(\forall yR(x,y) \rightarrow F(x)) : \\
F(x) \\
\forall yR(x,y)
\end{array} \]

4. **Identity**: Frege makes broad use of the identity sign \( \equiv \) to express sameness of semantic content between expressions. When placed between sentential expressions, \( \equiv \) acts as a biconditional, allowing the expression of claims such as:

\[ \begin{array}{c}
(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A) : \\
B \equiv A
\end{array} \]
Frege 1879 gives explicitly truth-functional semantics for the conditional and for negation:

If $A$ and $B$ denote judgeable contents, then there are the following four possibilities:
1. $A$ is affirmed and $B$ is affirmed;
2. $A$ is affirmed and $B$ is denied;
3. $A$ is denied and $B$ is affirmed;
4. $A$ is denied and $B$ is denied.

$B$ now denotes the judgement that the third of these possibilities does not obtain, but one of the other three does. (§5)

In Frege 1893, Frege goes further, and rather than just specifying truth conditions for negations and conditionals in terms of the truth conditions of their parts, holds that the connectives themselves express functions from truth values to truth values:

The value of the function $\rightarrow \xi$ shall be the False for every argument for which the value of the function $\rightarrow \xi$ is the True, and shall be the True for all other arguments. (§6, see also §12 for similar remarks about the conditional)

Frege also shows that disjunction and conjunction can be expressed using appropriate combinations of the conditional and negation. He distinguishes between inclusive and exclusive disjunction, with inclusive disjunction represented by:

![Inclusive disjunction](image)

and exclusive disjunction represented by:

![Exclusive disjunction](image)

Conjunction is represented by:

![Conjunction](image)

Frege’s proof theory is axiomatic – a collection of nine axioms combine with rules of modus ponens and universal substitution to give a logic which is sound and complete on its propositional and its first-order fragments. The propositional axioms are:
Frege shares with the standard modern-era logician a dismissive attitude toward modality – essentially his only remark on modality occurs in Frege 1879, in which he simply says:

The apodictic judgment differs from the assertory in that it suggests the existence of universal judgments from which the proposition can be
inferred, while in the case of the assertory such a suggestion is lacking. By saying that a proposition is necessary I give a hint about the grounds for my judgment. But, since this does not affect the conceptual content of the judgment, the form of the apodictic judgment has no significance for us. (§4)

11 Peirce and Peano

Although Frege’s treatment of connectives, as with his logical system in general, is a very close match for contemporary logic, it had only a minimal impact on other logicians at the time. In a brief review of the Begriffsschrift in 1880, Venn describes it as a “somewhat novel kind of Symbolic Logic, dealing much more in diagrammatic or geometric forms than Boole’s”, and concludes that its central logical observations are “all points which must have forced themselves upon the attention of those who have studied this development of Logic” and that Frege’s logic is “cumbersome and inconvenient” and “cannot for a moment compare with that of Boole”. Frege’s influence on contemporary logic begins to emerge only at the turn of the twentieth century, as Bertrand Russell discovers and is influenced by his work.

An earlier and powerful line of influence on Russell and twentieth century mathematical logicians more generally begins with Charles Sanders Peirce (1839-1914) and from there is passed through Ernst Schröder (1841-1902) to Giuseppe Peano (1858-1932). Peirce, beginning in Peirce 1870 and culminating in Peirce 1885, develops a quantified first-order logic that is formally equivalent to that set forth in Frege 1879. Peirce begins in a Boolean framework, which he then modifies extensively. He shares with Boole a fascination with tracing out elaborate relations between mathematics and a logical “arithmetic” – in Peirce 1870, he explores the logical significance of the binomial theorem, of infinitesimals, of Taylor and Maclaurin series, of quaternions, and of Lobachevskian spherical non-Euclidean geometry.\(^{24}\)

Peirce 1870 represents Peirce’s first attempts to extend Boole’s treatment of quantification to cover relational and hence polyadic quantification. This first attempt is complex and awkward, but by the time of Peirce 1885, a much simpler and more elegant system involving the use of subscripts to mark variable binding relations has been developed. The details of the treatment of quantification lie outside the scope of this piece, but several points of interest in the treatment of connectives emerge along the way. Peirce follows Jevons in recognizing the unsatisfactory nature of Boole’s addition operation, with its presupposition of class disjointness, and replaces it with what he calls a “non-invertible” addition operation symbolized by ‘+’. When interpreted as an operation on propositions, rather than classes, ‘+’ is then an inclusive disjunction.

\(^{24}\)He argues, for example, that when the parallel postulate is given its quaternion interpretation as the claim that the square of a vector is a scalar, and when terms are expressed using what Peirce calls “elementary relatives” in a way that models the arithmetic of quaternions, we find that the parallel postulate holds good only with the ultimate elements of space, so its logical equivalent holds good only for elementary relatives. (Peirce 1870, 366)
Peirce innovates more significantly by augmenting Boole’s uniform treatment of logic as a system of equations with the use of inequalities. Peirce uses the symbol ‘$\rightarrow <$’ as his symbol for inequality. When terms are interpreted as representing classes, ‘$\rightarrow <$’ then represents the subset relation. Peirce 1885 emphasizes the role of ‘$\rightarrow <$’ as a conditional connective. The conditional is taken primarily as a strict conditional:

The peculiarity of the hypothetical proposition is that it goes out beyond the actual state of things and declares what would happen were things other than they are or may be. … There can be no doubt that the Possible, in its primary meaning, is that which may be true for aught we know, that whose falsity we do not know. The purpose is subserved, then, if, throughout the whole range of possibility, in every state of things in which A is true, B is true too. (Peirce 1885, 186-187)

But Peirce then immediately converts this strict conditional into a material conditional by noting that “the range of possibility is in one case taken wider, in another narrower; in the present case it is limited to the actual state of things” (Peirce 1885, 187). Peirce then gives the truth conditions for the material conditional in terms of the truth values of antecedent and consequent, and uses these truth conditions to justify axioms for the conditional.

Peirce gives five axioms for the conditional:

1. $A \rightarrow B$
2. $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
3. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
4. $\neg A \rightarrow (A \rightarrow B)$
5. $((A \rightarrow B) \rightarrow A) \rightarrow A$

The fifth axiom has come to be known as Peirce’s Law; it is equivalent to the principle of the excluded middle, and like that principle, it fails in intuitionistic logic.

Peirce’s use of inequalities allows him to avoid the use of Boole’s indefinite class symbol $v$ in representing conditionals, and his treatment of quantification, including the use of explicit existential (‘$\Sigma$’) and universal (‘$\Pi$’) quantifiers allows him also to avoid the use of $v$ in representing indefinite claims, thus fully removing $v$ from the logic. Peirce notes the relation between existential and universal quantification, on the one hand, and (potentially infinitary) disjunctions and conjunctions, on the other hand.

In later work, Peirce developed what he called “existential graphs” – a system of logic based on diagrammatic, rather than linguistic, representations. The simplest system of existential graphs is the so-called alpha system. In the alpha system, the placement of two graphs next to each other represents the conjunction of their contents, and the “cutting”, or encircling, of a graph represents the negation of its content. The alpha system is thus primarily a minor representational reworking of standard propositional logic, although the choice of conjunction and negation as the sole logical operations is interesting in light of the later proof by Post 1921 that these two connectives are jointly functionally complete. Peirce’s beta system extends the alpha system to include
quantification. The beta system adds a novel sort of connective – variable coindexing is treated as a connective, by using a line to connect monadic predicates to represent the fact that the two predicates are predicated of the same object. Peirce’s gamma system then extends the logic to modality, by using a “broken cut” diagrammatic representation which represents the possibility of falsity. Although Peirce’s existential graphs had relatively little impact on the subsequent development of logic, they have in recent years been the focus of increased attention.

Ernst Schröder (1841-1902)’s 1895 work ‘Vorlesungen über die Algebra der Logik’ serves as a point of entry for the logic of Peirce and Boole into the continental European mathematical logic community. A standardized logical system, which serves as the framework in which much of the mathematical logic of the early twentieth century is done, is then produced by Peano. Peano 1893 inherits from Schröder an approach on which logic is treated as a system of classes. However, he adds to Boole’s system a symbol \( \varepsilon \) for the relation between individuals and classes, and allows a limited form of quantification over individuals – he allows variables to be subscripted to his conditional symbol, by which the universal quantification of the conditional with respect to those variables is represented. Like Frege, he uses a syntax that explicitly allows recursive embedding of connectives, and observes the definability of disjunction out of negation and conjunction. Unlike Frege, he does not give explicit truth conditions for his connectives. He takes the conditional to express deducibility of the consequent from the antecedent, rather than as a material conditional.

12 On to the Twentieth Century

As the nineteenth century draws to a close, the pieces are in place for the standard contemporary treatment of the connectives – the recursive syntax, the truth-functional semantics, and a well-formulated proof theory – but approaches vary considerably from logician to logician. Hugh MacColl (1837-1909) says:

Various logicians have adopted various symbols, each giving some reason founded on some mathematical analogy for his own special choice. Boole adopts \( P = 0 \) \( Q \) or \( P = vQ \); Pierce [sic.] takes \( P—<Q \); Schröder uses \( P = (Q) \); and there are many others; each writer, as I have said, justifying his choice on the ground of some real or fancied mathematical analogy. My own choice has been the symbol \( P : Q \) not (as has been erroneously supposed) on the ground of any analogy to a ratio or division, but simply because a colon symbol is easily formed, occupies but little space – two important considerations – and – though this is less important – because it is not unpleasing to the eye. (MacColl 1895, 494)

MacColl also anticipates two aspects of twentieth century work that had receded to the background in most modern-era work in logic: an intense interest in the logic of modality, and a willingness to experiment with non-classical logics. MacColl suggests that a two-valued logic be replaced with a three-valued one:

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In dealing with implications of the higher degrees (i.e. implications of implications) a calculus of two dimensions (unity and zero) is too limited, and that for such cases we must adopt a three-dimensional classification of our statements. We have often to consider not merely whether a statement is true or false, but whether it is a certainty, like $2+3=5$; an impossibility, like $2+3=8$; or a variable (neither always true nor always false), like $x=4$. (MacColl 1895, 496)

MacColl thus introduces the symbols $\epsilon$ for certainty, $\eta$ for impossibility, and $\theta$ for variability. These symbols play three separate roles for MacColl (with the roles often run together). They are names for semantic values, sentential operators indicating that a sentence has the name semantic value, and sentential constants picking out sentences which always have that semantic value. Thus MacColl will say that sentence $\alpha$ is of class $\epsilon$, will say that $(\alpha \beta)^\eta$ implies $\alpha^\eta \beta^\eta + \alpha^\theta \beta^\theta$, and will also say that the sentence $\alpha > \eta$, asserting that $\alpha$ implies $\eta$, is an impossibility.

MacColl’s three-valued treatment of modal notions anticipates Łukasiewicz’s 1920 three-valued system. However, a crucial difference is that MacColl does not take up in any systematic way the manner in which his three semantic values combine under various connectives. MacColl uses a superscript notation to indicate that a sentence has a particular semantic value – $\alpha^\eta$ means, for example, the impossibility of $\alpha$. Elsewhere in MacColl 1895, he also uses the superscript notion to indicate logical consequence, so that $\alpha^\eta$ means that $\beta$ implies $\alpha$. When using the superscript notation in this way, he investigates the analogies between exponentiation and his representation of implication, and endorses the principle that $(\alpha \beta)^\eta = \alpha^\eta \beta^\eta$. (This principle amounts to a rule of conjunction elimination – if $\gamma$ implies the conjunction of $\alpha$ and $\beta$, then it also implies each of the individual conjuncts.) When superscripts change to indications of semantic value, however, the question of the correctness of that principle is not re-opened. He endorses some other points of the analogy – he holds that $(\alpha \beta)^\eta = \alpha^\eta \beta^\eta$ (and that both represent the certainty of the impossibility of $\alpha$, indicating a willingness to consider iterated modals) – and he gives the specific instance that $(\alpha \beta)^\tau = \alpha^\tau \beta^\tau$, where $\tau$ is truth, but he does not raise the more general question.

The interactions between modal semantic values and conjunction and disjunction are thus not taken up systematically by MacColl, but he does make a systematic pronouncement on the modal-conditional interaction. Here he says:

No implication $\alpha : \beta$ can be a variable when …its antecedent $\alpha$ and consequent $\beta$ are both singulars; that is to say, when each letter denotes only one statement, and always the same statement, be it of the class $\epsilon$ or $\eta$ or $\theta$. Hence, $\alpha : \beta$ being synonymous with $\alpha \beta^{\eta}$ [$\beta^\eta$ is the negation of $\beta$] must be either = $\epsilon$ or else = $\eta$; it cannot be = $\theta$. (MacColl 1895, 503)

The view that conditionals, as themselves modal claims (the impossibility of the conjunction of the antecedent with the negation of the consequent), are either necessary or impossible, but never contingent, is an endorsement of the 4 axiom $\Box \alpha \rightarrow \Box \Box \alpha$ and the 5 axiom $\Diamond \alpha \rightarrow \Box \Diamond \alpha$ as basic modal principles.

MacColl closes his paper by suggesting that we might also want a nine-valued logic, in which the three modal values $\epsilon$, $\eta$, and $\theta$ are combined with three epistemic
values $\kappa$ ("known to be true"), $\lambda$ ("known to be false"), and $\mu$ ("neither known to be true nor known to be false"), to allow claims such as $\alpha^{\kappa}$ ("it is known that $\alpha$ is always true") and $\alpha^{\kappa}$ ("it is always true that $\alpha$ is known"). The resulting proliferation of options anticipates the proliferation of logical options the twentieth century would witness as the mathematical methods introduced in the nineteenth century came to fruition. The twentieth century development of logic lies outside the scope of this historical overview, but some major lines of investigation springing from the nineteenth century work include:

1. The emergence of metalogic in the twentieth century resulted in Post 1921’s proof of the functional completeness of (various subsets of) the standard connectives, and Sheffer 1913’s observation that the Sheffer stroke, or NOR, connective is by itself functionally complete. The development of the method of truth tables by Post and in Wittgenstein 1922 also shows the influence of an increased metalogical interest.

2. Modal logic, and the modal operators of necessity and possibility, becomes a topic of greatly increased interest. The work in Lewis 1918 and Lewis 1932 in characterizing a range of systems of modal logic did much to re-initiate interest in the area, and the subsequent development in Kripke 1959 and Kripke 1963 of possible worlds and frame-based semantics showed the considerable logical power of these systems.

3. Multivalued logics and treatments of the logical connectives became a topic of increased interest with the work of Post 1921 and Łukasiewicz 1920. Important subsequent work on multivalued logic occurs in Gödel 1932, Bochvar 1938, and Kleene 1938.

4. Nonclassical logics become a topic of intense interest in twentieth century work on logic. The intuitionistic logic of Brouwer 1907 and Heyting 1956 is the first major nonclassical system, and is then followed by the relevance and entailment logics surveyed in Anderson and Belnap 1975, and by other substructural logics. A central concern in much discussion of nonclassical logics is finding a treatment of conditionals which allows them to be properly integrated with other connectives in a given nonclassical system.

13 References


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*De Arte Disserendi*. in de Rijk 1967, 127–133.


