Substitutional interpretations of the quantifiers have succeeded in dealing with finite and even countably infinite domains, but meet their nemesis in non-denumerable universes. Any theory requiring such a universe—set theory, for example—has thus remained immune to substitutional interpretation. The author argues that a slight revision of substitutional semantics enables it to handle domains of any cardinality. He alters the standard valuation recursion by incorporating extensions of substitutional models. He then uses variants of the Koenig and Richard paradoxes to show that, though each parametric extension of the language may have only countably many parameters, there are uncountably many distinct parametric extensions. The author concludes by pointing out several other advantages of his new semantics.

Substitutional interpretations of the quantifiers have succeeded in dealing with finite and even countably infinite domains. In their traditional forms, however, these construals meet their nemesis in non-denumerable universes. Any theory requiring such a universe—set theory, for example—has thus remained immune to substitutional interpretation.

In this paper I shall argue that a slight revision of substitutional semantics enables it to handle domains of any cardinality. Though my version of substitutional quantification will diverge from the usual approach in certain respects, it will analyze the indeterminate reference of quantifiers and variables in terms of the determinate reference of names. It thus retains all the advantages of standard substitutional semantics, while vastly increasing that theory's power.

1. Substitutional Quantification and Uncountable Domains
I shall begin by defining a first-order language $L$ as containing a countable set of individual variables, $\{v_1, v_2, \ldots, v_n, \ldots\}$, a countable set of individual constants, $\{a_1, a_2, \ldots, a_n, \ldots\}$, a finite or count-
able set of $n$-ary predicates for each $n$, and the logical symbols '$\sim$', '$\&$', '$\exists$', '$(', and $\)$. To simplify, I shall treat identity as nonlogical, if it is present at all. Any individual constant or individual variable is a term. An $n$-ary predicate followed by $n$ terms is a formula; if $A$ and $B$ are formulas, then so are $\sim A$ and $(A\&B)$. If $A$ is a formula with one free variable $v_i$, then $(\exists v_i) A$ is a formula too. The set of formulas is the smallest set satisfying these conditions. Throughout this paper I shall speak, unless otherwise indicated, of languages having, or capable of being given, a first-order structure.

The usual referential semantics for such a language is familiar, using models, domains, denotation functions, etc. It specifies that an atomic formula is true on a model $M$ just in case the $n$-tuple of values of the terms, under the denotation function, satisfies the predicate. It proceeds to the remainder of the truth definition:

$$\begin{align*}
V_M(\sim A) &= T \text{ iff } V_M(A) = F; \\
V_M((A \& B)) &= T \text{ iff } V_M(A) = V_M(B) = T; \\
V_M((\exists v_i)A) &= T \text{ iff } V_M(A/v_i) = T \text{ for some term } t, \text{ where } M' \text{ differs} \\
&\text{from } M \text{ at most on the denotation of } t.
\end{align*}$$

Here $A/v_i$ results from substituting $t$ for all free occurrences of $v_i$ throughout $A$.

The last clause of the truth definition for $L$ characterizes the semantics as referential; it makes essential use of the denotation function. To obtain a substitutional interpretation, we alter the clause to the following:

$$V_M((\exists v_i)A) = T \text{ iff } V_M(Aa/v_i) = T \text{ for some constant } a_j$$

In other words, an existentially quantified formula is true just in case it has a true substitution instance in the language.

Once we have made this change, certain others become helpful. We no longer need to worry about the interpretation given formulas with free variables, so we can assume that only closed atomic formulas receive truth values in the basis clause. Furthermore, how they receive those values matters little. We might retain the standard referential account of truth for atomic sentences, or we might replace it with something else that takes closed atomic formulas into $\{T, F\}$. From the perspective of substitutional semantics, we will consider a “model” any mapping from closed atomic formulas into truth values. However we construe this mapping, the substitutional recursion will determine a truth value for all and only closed formulas. Moreover we can show, given a valuation function for the atomic case, that the recursion clauses determine a unique mapping of closed formulas of $L$ into truth values (Kripke, 1976).
I shall argue that we can provide a substitutional account of
the truth of any first-order theory, even those ostensibly commit-
ing us to uncountable domains. The substitutional approach I shall
advocate will differ in the formulation of its quantificational clause
from the above, more typical substitutional theory. Nevertheless,
it will retain all substitutional quantification’s most notable phil-
osophical features. The truth of quantified sentences will be defined
in terms of the truth of their instances; formulas with free variables
will receive no interpretation at all. Thus the reference of variables
and, in general, the quantificational apparatus will depend on the
reference of individual constants.

I propose to amend the quantificational clause of the substi-
tutional recursion:

\[ V_M((\exists v_1)A) = T \text{ iff there is a constant } \epsilon \text{ in some parametric ex-
tension } M' \text{ of } M \text{ such that } V_{M'}(Ac/v_1) = T. \]

I speak here of parametric extensions of ‘models’ rather than
languages: given an extension of L to L’ by way of the addition
of new constants, a model M’ of L’ extends M of L just in case
M’, restricted to the constants of L, is identical to M.

In most applications we would want to add to the above clause
that M’ must conform to the set of criteria determining the as-
ignment of truth values to atomic subformulas in M. In Kripke’s
terms, M will assign values in the basis clause according to some
metalinguistically defined predicate R; we want M’ to conform to
M by also employing R. We ordinarily want ‘There are flying
horses’, for example, to be false; the extension of M containing
‘Pegasus’ and assigning truth to ‘Pegasus is a horse’ and ‘Pegasus
flies’ we must therefore exclude from consideration when we con-
struct a ‘factual’ valuation.

But this talk of a metalinguistic predicate does not suffice to
guarantee that the recursion determines a unique mapping of closed
formulas of L into \{T, F\}. To accomplish that, let a parametric
model \( M = (\alpha, \Gamma) \) be an ordered pair consisting of a mapping
\( \alpha \) from atomic formulas of L into truth values and a set \( \Gamma \) of
mappings \( \beta \) from atomic formulas of parametric extensions of L
into truth values such that (a) \( \alpha \in \Gamma \) and (b) \( \beta \) restricted to the
constants of L is identical to \( \alpha \) (symbolically, \( \beta \upharpoonright L = \alpha \)). Say
that \( M' = (\alpha', \Gamma') \) of L’ is a parametric extension of \( M = (\alpha, \Gamma) \)
of L just in case (a) \( \Gamma' \subseteq \Gamma \) and (b) every \( \beta \in \Gamma \) such that \( \beta \upharpoonright L' = \alpha' \) is such that \( \beta \in \Gamma' \). This suffices to guarantee uniqueness
(Bonevac, 1985).

Intuitively, a parametric model consists of an assignment of
truth values to the atomic formulas of the language and a class
of extensions of that basic assignment. We may think of the basic assignment as proceeding according to certain rules (expressed by R); the class of extensions of this assignment tells us which names we can add to the language and, according to R, with which results. I shall refer to the extensions conforming to the rules of the basic assignment as admissible in M. The game class hence specifies the class of admissible assignments in any context.

A parametric model M' extends another, M, just in case M''s game class includes exactly those assignments admissible in M that extend M'''s basic assignment, α', which itself extends M's assignment α. I intend this concept to reflect formally an aspect of what happens when a speaker adds a sentence or set of sentences to a body of discourse. Think of α as the assignment to atomic sentences agreed upon by the speakers involved in the discourse. When a speaker initiates or continues the discourse, the basic assignment typically extends, say to α'. Additionally, the speakers initially carry on their discourse according to more or less precisely defined rules. Γ represents the set of admissible extensions of α relative to the operative discourse rules and "the facts of the matter". When α extends to α', the class of admissible extensions shrinks to include just those elements of Γ that extend α' unless the added information somehow changes the rules of discourse.

I am arguing that this parametric substitutional truth recursion can cope with nondenumerably infinite domains. The problem of uncountable domains arises because we have only denumerably many constants in the language. Any model with an uncountable domain must therefore be nonHenkin. But since the standard substitutional truth concept may appear to diverge from the classical notion on such models, substitutional quantification may distort our ordinary concept of truth whenever we intend our bound variables to range over a nondenumerable universe. Parametric substitutional quantification, however, avoids any inherent problems concerning nonHenkin models. Any objects unnamed in the language can acquire names in some parametric extension. So the concept of truth can conform to the standard concept even when we quantify over uncountable collections.

But this argument is open to a damaging objection. We have assumed that our language has only denumerably many constants. It seems reasonable to suppose that any language could have at most countably many constants. But then the totality of parametric extensions of our language might contain only denumerably many constants; we may not have enough names to go around, even in all the extensions of our language. Thus the premise on which the
above argument relies— that any object can be named in some parametric extension of the language—may be false.

Clearly, to argue that parametric substitutional quantification conforms to our usual concept of truth when we intend to quantify over uncountable universes, we must argue that any object in the universe can be named in some parametric extension. But we must take care to observe the distinction between extensions of models and extensions of languages. Even if we grant that we can extend the language in only countably many ways, we can extend any model M in uncountably many ways. For suppose that we can extend L to include $\aleph_0$ new names; then there will be $\aleph_0$ new atomic sentences, any subset of which an extension of M may designate as true. We can extend M, therefore, in at least $2^{\aleph_0}$ ways. This suffices for domains of the cardinality of the real numbers.

We can account for even larger domains if we can extend a language in more than countably many ways. If we can extend L in $2^{\aleph_0}$ ways, for example, we can extend any model in $2^{2^{\aleph_0}}$ ways, enough to capture domains of the cardinality of the functions on the reals.

In any case, it may seem that the above objection misses the point. Suppose that we have only countably many constants. If we do not fix their denotations in advance, we can name any real number simply by introducing a constant for it (though we cannot do this for all reals at once). So of course every real number can be named. I think this intuition is essentially correct. But I shall set it aside, because I want to convince even those with contrary intuitions. One might argue that, to name a real number by introducing a constant for it, one must be able to specify which real one has in mind. But we cannot specify every real number within a first-order theory (Quine, 1976). Moreover, this strategy requires us to treat the reals as objects standing in referential relations to certain constants. It forces us to view mathematics as describing an uncountable universe of objects. Anyone unwilling to accept second-order specifications of real numbers or talk of the reals as constituting a nondenumerable set of objects will demand a more complex line of argument. I shall thus assume, throughout most of what follows, that we have fixed the "denotations" of constants in advance, perhaps in the construction of the constant itself.

To argue that any language has more than countably many extensions, therefore, I need to argue either that (a) some parametric extension of the language contains nondenumerably many constants, or that (b) though each parametric extension of the language has only countably many constants, there are uncountably
many parametric extensions. Since the union of countably many countable sets is still countable, we must have either some uncountable set or an uncountable number of sets to obtain an uncountable totality.

Can a parametric extension of some language L contain non-denumerably many constants? Some philosophers hold that it can, or, at least, that we cannot prove that it cannot. Syntactically, constants are usually generated by a recursive rule, as in L; the recursiveness of the rule for construction guarantees that we can have at best countably many constructs. But philosophers who characterize constants in terms of their function typically deny that rules for construction need be recursive, or, indeed, that any rules need be specified at all (Sicha, 1974, pp. 73-74, 111; 1978, pp. 257, 260, 271). Others, in contrast, hold that no language the constants of which are not generated by a recursive mechanism from a finite number of primitives is learnable.3 But certainly there is no absurdity in the assumption that a language might contain more than countably many constants. Indeed, a well established logical metatheory exists for such generalized first-order languages (Mendelson, 1979, pp. 95-103).

Consequently, we face the problem: can human beings learn and use, from a practical point of view, a language with non-denumerably many constants? Davidson and his followers have answered this question by assuming that humans, at best, are Turing machines. Humans’ cognitive capacities prevent them from dealing with any language not recursively generated from a finite number of primitives. Since I have no idea how to decide whether humans have any cognitive capacities that go beyond those of a Turing machine, I shall leave this question for another occasion. I shall instead focus on option (b).

2. How many Extensions?

I claim that any language L has uncountably many parametric extensions. My argument for this assertion, if it succeeds, will justify my thesis that parametric substitutional quantification can handle domains of cardinalities greater than $2^{\aleph_0}$ without distorting our ordinary concept of truth.

I shall present several arguments to demonstrate that in fact a language has non-denumerably many extensions. My arguments will fall into two classes. In this section, I shall argue that the assumption that there are only countably many extensions of any language leads to paradoxes; resolution of the paradoxes yields arguments showing that there must be uncountably many exten-
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Throughout these arguments I shall assume that we take a Davidsonian approach to languages. I shall refer only to linguistic systems that can be generated recursively from a finite number of primitives. In the next section I shall argue that several commonly adopted ways of understanding language imply that any language can be extended in nondenumerably many ways. Throughout these arguments, I shall speak of examples of nondenumerable collections and theories having them as intended interpretations. I shall most often assume that we have in mind an elementary theory of the real numbers, with the predicates, constants, etc., being given their usual interpretations.

Suppose that a language has only denumerably many distinct extensions. Suppose further that our theory, in language L, has the reals as its intended model. Then, since each extension contains names for at most countably many reals, nondenumerably many others are essentially anonymous, i.e., have no names in any extension of L. Assume that the reals can be well-ordered. Then in any subset of the reals there is some first member. Consider the subset consisting of just essentially anonymous reals. It, too, must have a first member. But we may specify this member as "the first essentially anonymous real". This, of course, is not a name but a description; the distinction is far from trivial, for if descriptions are admitted into the substitutional truth characterization, we can no longer show that the class of true sentences has been uniquely specified by the recursion (Kripke, 1976, pp. 331-332, 368). Nevertheless, we may convert the description to a name by hyphenating spaces and treating the sequence, logically, as unanalyzed. 'The-first-essentially-anonymous-real' thus serves to name what, by assumption, cannot be named, giving us a contradiction.

This argument derives a contradiction from the assumption that a language may be extended in only countably many ways, but it uses the axiom of choice in the derivation. One might be tempted to take the argument as refuting not the countability assumption but the choice axiom. The argument would be stronger, therefore, if we could remove its dependence on the existence of a well-ordering of the reals.

To do this, assume once again that L, a language for a theory of the real numbers, can be extended in at best countably many ways. It follows that denumerably many reals have names in L or an extension of L; the others are essentially anonymous. Enumerate the nameable reals, and imagine them expressed in standard decimal form. Now form a real not belonging to this set. Let \( \rho \) be the numeral in the \( n \)th decimal place of the \( n \)th member of the
set of nameable reals. Form a number having 0 as its integral portion and having in its \( n \)th decimal place \( p + 1 \), if \( p \) is not 8 or 9, and otherwise 1. This number is not a member of the set of nameable reals. Nevertheless, we have a name for it:


This number appears to be both nameable and essentially anonymous, a contradiction. Since the assumption that \( L \) has only countably many extensions leads to a contradiction, it must be false. \( L \) must be parametrically extendible in nondenumerably many ways.

These arguments are, in essence, the paradoxes of Koenig and Richard (Koening, 1967, Richard, 1967). They thus lie vulnerable to a serious objection. The paradoxes are certainly striking, one might charge, but they demonstrate nothing about the countability or uncountability of parametric extensions. They seem plausible because they employ certain illegitimate locutions, formulations, concepts, etc. Thus, in response to my arguments above, we should attempt to discover the fallacy implicit in them; we should not throw out the assumption from which the arguments started.

Before evaluating this objections, I shall present an argument capable of serving as a paradigm for arguments for uncountability: Georg Cantor’s proof of the nondenumerability of the reals. We begin by assuming that the reals are denumerable. If we enumerate them according to their decimal expansions, we can construct a real, as above, that does not belong to the list. To make the argument more explicit:

(1) \( \bar{\aleph} \leq \aleph_0 \) (assumption)
(2) \( \aleph \) is enumerable (1)
(3) (\( \exists \alpha \) (List(\( \alpha \)) \& \( \alpha \) enumerates \( \aleph \)) (2)
(4) List (\( \alpha_0 \)) \& \( \alpha_0 \) enumerates \( \aleph \) (3)
(5) (\( \exists x \) (\( x \in \bar{\aleph} \& x \notin \alpha_0 \)) (Construction)
(6) \( \sim (\alpha_0 \text{ enumerates } \aleph) \) (5)

Here \( \bar{\aleph} \) the cardinality of \( \aleph \), the set of all real numbers. Though Cantor’s proof has stirred some controversy, contemporary mathematicians almost universally accept it. I shall assume its validity throughout the following.

To explicate the structure my arguments given above share (where \( D \) is the set of nameable reals):

(1) \( D \leq \aleph_0 \) (assumption)
Clearly we cannot challenge the moves to (2), (3), (4), or (6). That leaves (1) and (5). Koenig rejects (5); he takes his argument to show that the reals cannot be well-ordered. Richard, too, balks at (5), rejecting the claim that \( x \in D \).

But the construction here in (5) is virtually identical to Cantor's. We enumerate the set of nameable reals according to their decimal expansions, then construct a real not on the list using exactly Cantor's method. In constructing this real, of course, we construct two names of it: one a standard decimal name, which is infinite, and another which is finite. We construct the finite name by referring to the set of nameable reals. Cantor, in contrast, needs to refer only to the set of all reals. We can distinguish the two constructions only by denying the legitimacy of reference to 'nameable reals'.

Bertrand Russell develops the standard approach to the Koenig and Richard paradoxes, which I shall call the restrictive approach; it adopts precisely this strategy. Russell finds that all the paradoxes make use of self-reference. In the Koenig and Richard paradoxes, nameability and definability are elements of names and definitions. We cannot refuse to accept all such self-reference, for often it is harmless. Russell concludes that "all definitions" and "all names" are illegitimate notions. We ought to speak of definability or nameability relative to a given set of linguistic or conceptual resources rather than of definability and nameability tout court. When we do this, we find that "definable in terms of ideas of the class N" is not definable in terms of ideas of the class N' (Russell, 1967, pp. 154, 155, 166). According to the restrictive approach to the paradoxes, quantification with respect to language variables is illegitimate. We preclude paradoxical conclusions by denying that the expressions 'is definable', i.e., 'is definable in some language or other', and 'has a name', i.e., 'gas a name in some language or other', are well-formed.

The restrictive approach to the nameability paradoxes creates problems for the arguments I am advancing. Essentially anonymous reals cannot be named in any linguistic system; nameable reals have names in some linguistic system. Without these notions, my arguments cannot get off the ground.

Luckily, Ernst Zermelo presents another approach, which I shall term expansive:
‘Finitely definable’ is not an absolute notion but only a relative one, and it is always related to the chosen language, or notation. The conclusion that [the set of] all finitely definable objects must be denumerable holds only if one and the same system of signs is to be used for all of them, and the question whether a single individual can or cannot have a finite designation is in and of itself meaningless, since to any object we could, if necessary, arbitrarily assign any designation whatever.5

Zermelo thus agrees with Russell that nameability and definability must be understood as relative to a specified language. He concludes that, though only denumerably many objects are nameable within any one linguistic system, we cannot assume that the set of nameable objects is countable if we consider more than one such system.

Zermelo, using language reminiscent of Russell, says that absolute nameability and definability are meaningless, but he explains that “to any object we could, if necessary, assign any designation whatever.” Zermelo asserts that all objects are nameable in some linguistic system or other. But this assertion makes sense only if quantification with respect to language variables is permitted.

How, then, does Zermelo avoid the paradoxes? Koenig and Richard begin with a language having denumerably many names or definitions. They discuss the problem of which, among a non-denumerable set of objects, have names. Either they mean ‘have names in the system’ or ‘have names in some system or other’. If the former, then only denumerably many have names in the system, so (1) is justified. We thus construct the appropriate name or definition, e.g., ‘the first essentially anonymous real’, finding that an object having no name is nameable after all. But, on this alternative, there is no contradiction, for we have shown only that some object without a name in some particular language can be named in some language or other. Thus, in step (5) we cannot justify the assertion that \( x \in D \). We have extended our original language to name an object beyond the original language’s grasp, but this is hardly problematic. The moral of the paradox is that, as Russell claims, ‘the first essentially anonymous real’ is not a name of the original system.

Suppose that instead Koenig and Richard mean to quantify with respect to language variables, and so intend by ‘is nameable’ ‘has a name in some language’. Then how many reals are nameable? Koenig and Richard would say denumerably many, concluding that uncountably many others must be essentially anonymous. But Zermelo’s insight is that, on this construal of the argument, the paradoxical conclusions show that their answer is
incorrect. The construal of 'nameable' that justifies (5) gives us reason to reject (1). We can quantify with respect to language variables without contradiction, so long as we do not assume that nameability in a given system is closely analogous to nameability in some system or other. In particular, we must not presume to enumerate all possible names or definitions. Zermelo's analysis is expansive in the sense that he regards the paradoxes as showing not that restrictions need to be imposed but that our linguistic resources are far more powerful than many have thought: in particular, that more than countably many reals are nameable.

Both the restrictive and expansive approaches, then, avoid the Koenig and Richard paradoxes. But only under the expansive approach do my arguments based on the paradoxes succeed. I wish to contend, naturally, that we should adopt the expansive approach. Zermelo's response to the nameability paradoxes is mandated by a principle of methodological conservativism. Both techniques eliminate the paradoxes, but Russell's has restrictions which depart substantially from our original, intuitive account of nameability. To put the point more strongly, since the expansive approach handles the paradoxes, the demand for avoiding contradiction cannot justify the Russellian restriction on quantification with respect to language variables. But it appears to have no other justification.6

3. Linguistic Types

I am arguing that, though each linguistic system contains at most countably many names, each can be extended in more than countably many ways. One might summarize my view as the assertion that there are uncountably many possible names. But since talk of possible linguistic objects sounds obscure, I prefer to say that there are nondenumerably many linguistic types. We may think of a language as containing at most denumerably many linguistic items, each of which "instantiates" a linguistic type. Though a given language can instantiate at most countably many types at once, it can be extended in more than countably many ways; that is, the language together with all its extensions can, in principle, instantiate nondenumerably many types.7 But what is a linguistic type? Equivalently, how are we to classify linguistic items? If we could answer these questions, we would be in excellent position to discover how many types there are. But here I shall not try to argue for any particular conception of types. Instead I shall contend that several commonly held attitudes toward linguistic types imply that there are nondenumerably many of them.

We might think, initially, that names are simply sign designs.
From this purely syntactic perspective, how many name-types are there? If we limit ourselves to two dimensions and to some finite region, then there will be as many name-types as subsets of points of that region. Since there are nondenumerably many points in any such region, there must also be nondenumerably many subsets of points and, so, nondenumerably many name-types. Perhaps only denumerably many of these are humanly recognizable within the confines of a linguistic system. Nevertheless, it seems reasonable to suppose that any sign design could be humanly distinguishable within some linguistic system or other, and this suffices to yield nondenumerably many name-types.

If we attempt to individuate name-types semantically, however, the situation becomes more complicated. We might think that an account of the reference of names should precede a theory of the indeterminate reference of quantification; the account we give of determinate reference may take a variety of forms.

First, consider the view that we distinguish linguistic types, primarily, on the basis of language entry roles. Briefly, the language entry role of a given linguistic item is described by indicating under what conditions that item is tokened in response to environmental stimuli. For some bits of language, entry roles are very difficult to characterize, and tell us little about the linguistic functioning of the term. But with regard to names, the bits of language we care about, language entry roles seem quite significant. Describing the entry role of a name is describing under what stimulus conditions we are willing (and correct) to token it; it is, in other words, giving the behavioral criteria for the name’s applicability. Now suppose that we want to distinguish name-types solely on the basis of language entry roles. How many name-types will result? First we must determine how many different patterns of environmental stimulation—how many different situations, under which we might or might not properly token the name—we are willing to recognize. Suppose that we are willing to recognize countably many distinct environmental situations. It seems that we could count any subset of this denumerable collection as justifying the application of the name is question. But then any subset would characterize a linguistic type; since there are uncountably many subsets of any denumerably infinite set, there must be uncountably many types.

Someone may object that many of these subsets will not characterize anything we would ordinarily consider an object. We might, however, speak of predicates or sentences rather than names. Here it is even clearer that we could count any subset of the set of environmental conditions as characterizing a linguistic type. But
if we conclude that there are uncountably many predicate- or sentence-types, then by considering any tokening of a predicate or a sentence a tokening of a name of the relevant linguistic type, we can conclude as well that there are uncountably many name-types.

Second, suppose that we distinguish linguistic types on the basis of satisficalional roles. We might decide that names are to be described by specifying which atomic sentences containing them are true. In short, we might characterize name-types according to their roles in satisfying or failing to satisfy atomic predicates. Suppose that our language has countably many unary atomic predicates. A name might satisfy, or fail to satisfy, any of them, quite independently of whether or not it satisfies the others. In short, a name might satisfy any subset of the countable set of unary predicates. Obviously, then, there must be uncountably many name-types, for there are uncountably many such subsets.9

Third, suppose we attempt to characterize types in terms of the meanings or intensions of their instances. Name-types, of course, appear to have no associated meanings. But for the moment, I shall neglect this complication, and speak of linguistic types in general. An additional problem arises: what is a meaning? If we take meanings as unanalyzed objects, then we can make no progress. We might, however, adopt one of several more sophisticated views about meanings. (1) Say that we take meanings as corresponding roughly to truth conditions (Gottlieb, 1980). How many truth conditions could pertain to a certain sentence? Presumably as many as could be expressed in an appropriate metalanguage. Each such metalanguage would have only countably many expressions, given our earlier Davidsonian assumptions. But each would also be extendible to allow for the formulation of additional truth conditions. In how many ways could a metalanguage be extended? I have been arguing that one can be extended in nondenumerably many ways. Since it is hardly fair to use the desired conclusion as a premise, we must refrain from drawing any implications from the view of meanings as truth conditions. That approach to linguistic types remains neutral with regard to the cardinality of the possible extensions of a language.

(2) We might view meanings as corresponding to what David Kaplan has called transworld heir lines (Kaplan, 1979). Using Hintikka’s simpler terminology, world lines are mappings from possible worlds into objects, if we are speaking about terms or predicates, and into truth values, if we are speaking about sentences. How many mappings of possible worlds into objects could there be? Often it is thought that there must be uncountably many possible worlds, so the answer, if we agree, must be uncountably many.
But this holds even if we restrict ourselves to denumerably many possible worlds. How many distinct objects could there be? Let’s say, conservatively, finitely or countably many. Then we have mappings from a denumerable set into a finite or denumerable set. In either case, there are nondenumerably many such mappings. If we view meanings or intensions as corresponding to world lines, then, we must conclude that there are uncountably many of them.

(3) Suppose finally that we construe meanings as corresponding to, or, more strongly, as being functional classifications (Sellars, 1974). How many functional classifications can we have? Presumably as many as there are functional classes, i.e., as many as there are linguistic functions. How can we characterize the function of a linguistic item? We might characterize it in terms of its relation to extralinguistic conditions (in Sellarsian language, in terms of its language entry and exit roles) or in terms of intralinguistic conditions (in terms, i.e., of its inferential or satisfactional role). Probably our account of types will take both sorts of considerations into account. In any case, I have argued that, whether we characterize linguistic types in terms of language entry roles or in terms of inferential (or, in the case of name-types, satisfactional) roles, nondenumerably many linguistic types emerge. Thus a view of meaning as functional classification also implies that there are nondenumerably many possible meanings and, therefore, nondenumerably many linguistic types.

(4) We might, as Jon Barwise and John Perry suggest, take meanings as sets of supporting situations (Barwise, 1981) or as functions from discourse situations and connections to interpretations (Barwise and Perry, 1981). If we suppose, as seems reasonable, that there are at least countably many situations of each kind, then there are uncountably many meanings.

I do not mean to suggest that this survey of attitudes toward linguistic types is exhaustive. I do want to point out, however, that the view that there are uncountably many linguistic types, far from being patently absurd, is implicit in many commonly held theories of language. My arguments in the previous section thus play an unsuspected role; they reinforce what is already a presumption of many approaches to the philosophy of language. Perhaps we should hold the number of linguistic types nondenumerable until proven otherwise.

4. Implications

The parametric substitutional interpretation of the quantifiers I am advocating has a number of advantages. It allows for greater flex-
ibility than ordinary, objectual quantification, in that it does not commit us to speaking of reference to objects. The atomic sentences are assigned a truth value somehow, but in the general case we do not care about the details. We may assign truth according to satisfaction by objects of open sentences, but in other contexts we may assign truth in other ways. We can thus incorporate non-denoting terms; ‘Pegasus is a flying horse’ may receive a truth value quite apart from considerations of denotation and satisfaction. Furthermore, since we may allow truth to depend on names, rather than objects alone, we may use substitutional quantification in languages with opaque contexts. My parametric semantics therefore preserves the most significant advantages of substitutional interpretations in general.

Parametric semantics also has many advantages over usual substitutional approaches, in addition to its facility with nondenumerable domains. First, given the usual definitions of logical concepts, first-order logic is compact only if the quantifiers are read objectually. Consider the set of formulas

$$\{ \sim Fa_1, \sim Fa_2, \ldots, \sim Fa_n, \ldots, (\exists v)Fv \}$$

where $a_1, \ldots, a_n, \ldots$ are all the individual constants in the language. In standard metatheory, this set is consistent. But it is inconsistent when the quantifier is read substitutionally; the existentially quantified formula would have to be true without having any true substitution instances. On a substitutional reading, an existentially quantified formula is logically equivalent to an infinite disjunction of its instances (Dunn, 1968, Thomason, 1965). But if we interpret the quantifiers substitutionally in the parametric sense, we restore consistency. Consider the extension of our language which adds a single constant $b$. Assume that ‘$Fb$’ is true, but that ‘$Fa_i$’ through ‘$Fa_n$’, and so on, are false. Then all the statements in the set come out true. In short, the problem arises because the set contains formulas which jointly have occurrences of every constant in the language. Since there is always a parametric extension that adds a new constant to the language, our parametric quantification clause guarantees that no set of formulas can give rise to a problem of the kind Dunn, Belnap, and Thomason envision.

Parametric substitutional quantification similarly eliminates the problem of objects without names in non-Henkin models. Traditional substitutional quantification threatens to deviate from our usual account of truth with regard to sentences the truth or falsehood of which depends upon the existence of some unnamed object. Parametric substitutional quantification, however, allows us to add
names to the language, by counting quantificational sentences true only if there is some parametric extension of the language in which they have a true instance. Objects, consequently, may be unnamed in L, but named in some parametric extension of L. If we can, in principle, name any object in some parametric extension of the language, then we may assign quantified sentences truth values in a way that corresponds precisely to our ordinary understanding of truth. Whether an object has a name in the language as it currently stands makes no difference so long as we can extend the language to include such a name.

Additionally, parametric semantics eliminates the problem Quine and Weston present of theories which cannot accept a substitutional interpretation (Weston, 1974, p. 361; Quine, 1969, pp. 64-66). We may have a theory in which an existential quantification is true even though all its instances are false. But when we employ parametric substitutional semantics, this presents no problem: the existentially quantified sentence is true because it has a true instance in the language or one of its extensions. The universe may contain a nameless object, to put the problem in referential terms, but so long as the object is nameable in some parametric extension of the language we are not forced to interpret the quantifier objectively. Thus Quine and Weston cannot conclude that some theories decide in favor of referential quantification without reference to any background theory. Unlike traditional substitutional approaches, parametric substitutional semantics can interpret such theories without difficulty.

Moreover, parametric substitutional quantification reflects a usual and highly intuitive paradigm concerning the treatment of names. Speaking formally, we most often specify a language (such as L) by specifying its predicate letters, variables, logical symbols, and constants. But in dealing with natural languages we take quite a different approach to names. What are the names available in English? in French? in Urdu? These questions have no determinate answers. We generally do not find names in the dictionary. And giving a baby a name never before used does not count as changing the language. In short, we generally treat natural languages as if names were not an integral part of them. The stock of names in the languages may contract, expand, etc., without the languages themselves changing. Traditional substitutional interpretations have taken treatments of names in formal languages seriously, but have ignored the more usual situation. Parametric substitutional semantics, in contrast, allows that the stock of names may vary without affecting the truth values of sentences that seem to be independent of any assertions about names.
Finally, parametric substitutional quantification extends the power of substitutional interpretations. Set theory cannot accept a traditional substitutional interpretation unless we are willing to restrict our universe to predicative sets (Parsons, 1971, p. 236). Substitutional set theory thus seems to be restricted to a countable universe. But if we use a parametric substitutional semantics for set theory, we might in principle be able to account for the existence of nondenumerably many sets, even impredicative ones (Bonevac, 1983). Parametric substitutional quantification thus brightens the prospects for the ontological utility of a nonreferential semantics.

Some philosophers hold that substitutional semantics is ontologically neutral; others, that it involves an ontological commitment to linguistic items. Quine holds that its ontological commitments are relative to a background language in which the substitutional theory is given an objectual translation. I shall not attempt to adjudicate this issue here. In order to say anything about the ontology of substitutionally interpreted discourse, however, we need a theory concerning the ontological commitments of theories interpreted in ways other than the usual, referential manner. This in turn requires an explication of the link between objectual and substitutional quantification in both formal and natural languages.

References


**Notes**

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I shall call a model *Henkin* just in case every object in its domain has a name in the language. See (Leblanc, 1976, p. 311).


My arguments will demonstrate only that there are at least 2 \( ^{\omega} \) extensions of a language; though this suffices for a theory of functions of real numbers, since we thus obtain 2\(^{^{\omega}} \) \( ^{\omega} \) model extensions, other theories have even larger intended domains. Nothing I shall say, of course, precludes parametric substitutional semantics from dealing with such theories.

See (Zermelo, 1967, p. 192n). Zermelo credits this analysis to Gerhard Hessenberg.

One might argue that, since other paradoxes, e.g., the liar, require restrictions on quantification with respect to certain sorts of variables, uniformity of treatment might
constitute such a justification (Burge, 1979). In my opinion this would work only if the other paradoxes mandated the same restrictions on the same variables as were proposed to handle the nameability paradoxes. But they pertain to truth rather than to nameability.

7Wilfrid Sellars develops the concept of a linguistic type in (Sellars, 1967). Obviously my claims are coherent only if there are uncountably many linguistic types. Of course, this does not mean that I am committed ontologically to a nondenumerable infinity of types, though I might be. See (Bonevac, 1982, pp. 152-153). My arguments involve an *ostensible* or *prima facie* commitment to types, but these commitments are, in principle, rebuttable. In any case, these ontological matters do not affect the points I wish to make in this section.

8For a characterization of language entry roles, and how they function in a fuller theory analyzing linguistic behavior, see (Sellars, 1974, p. 123).

9Hence a first-order theory allows us to distinguish all reals, whether or not we can specify them. (Quine, 1969).

10We need at least two. If there are nonenumerably many, so much the better. Of course, if there are fewer than countably many objects, our original problem dissolves: parametric (or even standard) substitutional semantics has no problem with countable domains.